

PARTIAL FUNCTIONS

A partial function $f: \Sigma_1^* \rightarrow \Sigma_2^*$ is computed by M if whenever M is run on input word w , it halts in the halt state with output word $f(w)$.

A partial function is computable, if there exists a TM that computes it.

The identity function $\text{id}_{\Sigma^*}: \Sigma^* \rightarrow \Sigma^*$ is trivially computable.
 $w \mapsto w$

Suppose $f: \Sigma_1^* \rightarrow \Sigma_2^*$ and $g: \Sigma_2^* \rightarrow \Sigma_3^*$ are computable.
Then so is $g \circ f: \Sigma_1^* \rightarrow \Sigma_3^*$.

$$(g \circ f)(w) \cong \begin{cases} g(f(w)) & ; f(w) \downarrow \text{ and } g(f(w)) \downarrow \\ \uparrow & ; \text{otherwise} \end{cases}$$

Kleene equality:
lhs $\downarrow$ iff rhs $\downarrow$
if $\downarrow$, both sides are equal

$\uparrow$... undefined
 $\downarrow$... defined

PARTIAL FUNCTIONS ON NATURAL NUMBERS

What does it mean for a partial function $f: \mathbb{N} \rightarrow \mathbb{N}$ to be computable?

We can choose binary representation with leading zeros.

A partial function $f: \mathbb{N} \rightarrow \mathbb{N}$ is **computed** by M if, for any $w \in \{0,1\}^*$, if we run M on w , then we get a word w' as output, such that $\gamma_M(w') = f(\gamma_M(w))$.

A partial function $f: \mathbb{N} \rightarrow \mathbb{N}$ is **computable**, if there exists a TM that computes it.

$$\begin{array}{ccc} \gamma_M: \{0,1\}^* & & w \\ \downarrow & & \downarrow \\ \mathbb{N} & & \sum_{i=0}^{|w|-1} w_{|w|-i-1} 2^i \end{array}$$

Alternatively, we can choose binary representation without leading zeros:

$\varepsilon, 1, 10, 11, \dots$

$$\begin{array}{ccc} \gamma'_M: \{0,1\}^* & & w \\ \downarrow & & \downarrow \\ \mathbb{N} & & \sum_{i=0}^{|w|-1} w_{|w|-i-1} 2^i \end{array}$$

A partial function $f: \mathbb{N} \rightarrow \mathbb{N}$ is **computable** (w.r.t. γ'_M) if there exists computable $g: \{0,1\}^* \rightarrow \{0,1\}^*$, such that for all $n \in \mathbb{N}$ and words w where $\gamma'_M(w) = n$, there is $f(n) = \gamma'_M(g(w))$.

$\rightrightarrows$... partial function
 $\rightrightarrows$... partial surjective function

PARTIAL FUNCTIONS ON OTHER SETS

A **representation** of a set X by words over Σ is a surjective partial function $\gamma: \Sigma^* \rightharpoonup X$.

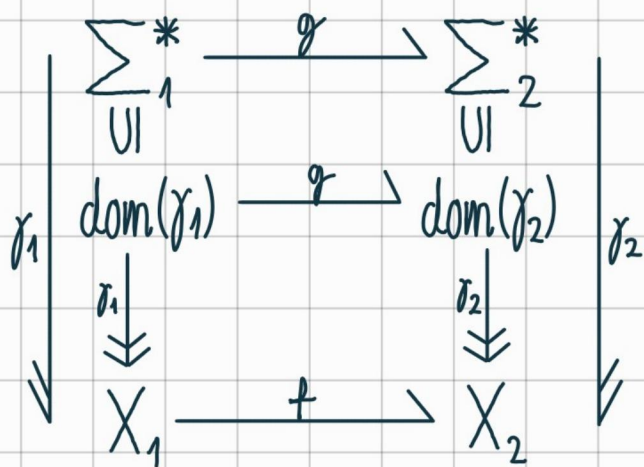
Note that X must be computable.

Definition: A **computable** partial function $f: X_1 \rightarrow X_2$ w.r.t. representations $\gamma_1: \Sigma_1^* \rightharpoonup X_1$ and $\gamma_2: \Sigma_2^* \rightharpoonup X_2$:

There exists a computable partial function $g: \Sigma_1^* \rightarrow \Sigma_2^*$, so for any $x \in X_1$, and any $w \in \Sigma_1^*$ with $\gamma_1(w) = x$ (w is γ_1 -name of x):

- $g(w) \downarrow \iff f(x) \downarrow$
- $f(x) \downarrow \implies f(x) = \gamma_2(g(w))$

($g(w)$ is γ_2 -name of $f(x)$)



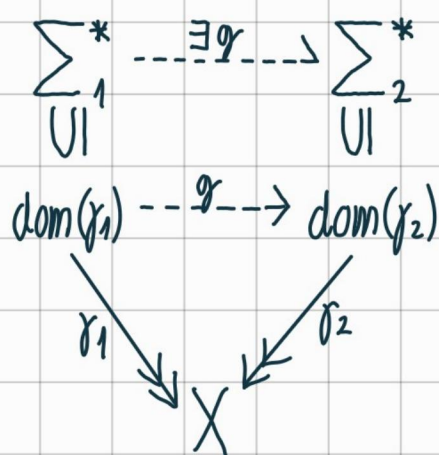
For any representations $\gamma_1: \Sigma_1^* \rightharpoonup X_1$, $\gamma_2: \Sigma_2^* \rightharpoonup X_2$, $\gamma_3: \Sigma_3^* \rightharpoonup X_3$:

- The identity function $\text{id}_{X_1}: X_1 \rightarrow X_2$ is computable.
- If $f: X_1 \rightarrow X_2$ and $g: X_2 \rightarrow X_3$ are computable (w.r.t. the above representations), then $g \circ f: X_1 \rightarrow X_3$ is computable (also w.r.t. the above representations).

Our two computability definitions for $f: \mathbb{N} \rightarrow \mathbb{N}$ give the same notion of computable partial function on $\mathbb{N}$:

- $\gamma_{\mathbb{N}}$ and $\gamma'_{\mathbb{N}}$ are equivalent representations ($\gamma_{\mathbb{N}} = \gamma'_{\mathbb{N}}$).
- The notion of computable partial function is invariant under equivalence of representations.

What does it mean for two representations $\gamma_1: \Sigma_1^* \rightharpoonup X$ and $\gamma_2: \Sigma_2^* \rightharpoonup X$ to be equivalent?



$\gamma_1 \equiv \gamma_2$ if $\text{id}_X: X \rightarrow X$ is $(\gamma_1 \rightarrow \gamma_2)$ -computable and $(\gamma_2 \rightarrow \gamma_1)$ -computable.

Proposition: If $\gamma_1 \equiv \gamma'_1$ are equivalent representations of X_1 and $\gamma_2 \equiv \gamma'_2$ are equivalent representations of X_2 , then $f: X_1 \rightarrow X_2$ is $(\gamma_1 \rightarrow \gamma_2)$ -computable iff it is $(\gamma'_1 \rightarrow \gamma'_2)$ -computable.

PRODUCT CONSTRUCTION ON REPRESENTATIONS

How should we define computability of $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$?

- We can use a 2-tape TM.
- We can use a bijection $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$.

- We can find a representation for the domain $\mathbb{N} \times \mathbb{N}$.

All approaches are equivalent.

We will elaborate on the 3rd approach.

Suppose we have k representations $\gamma_1: \Sigma_1^* \rightarrow X_1, \dots, \gamma_k: \Sigma_k^* \rightarrow X_k$.

We define a representation for the set $X_1 \times \dots \times X_k = \{(x_1, \dots, x_k); \forall 1 \leq i \leq k: x_i \in X_i\}$:

$$\Sigma := (\Sigma_1 \cup \dots \cup \Sigma_k) \oplus \{, \}$$

$$\gamma = (\gamma_1 \times \dots \times \gamma_k): \Sigma^* \rightarrow X_1 \times \dots \times X_k$$

$$\text{dom}(\gamma) := \{w_1, \dots, w_k; \text{ each } w_i \in \text{dom}(\gamma_i)\}$$

$$\gamma(w_1, \dots, w_k) := (\gamma_1(w_1), \dots, \gamma_k(w_k))$$

So, a partial function $f: \mathbb{N}^k \rightarrow \mathbb{N}$ is **computable** if it is $((\gamma_{\mathbb{N}} \times \dots \times \gamma_{\mathbb{N}}) \rightarrow \gamma_{\mathbb{N}})$ -computable.

DECIDING SUBSETS

Suppose $\gamma: \Sigma^* \rightarrow X$ is a representation.

Suppose $A \subseteq X$ is any subset.

We say that A is **decidable** w.r.t. γ if there exists a TM for language recognition that for any $w \in \text{dom}(\gamma)$:

- If $\gamma(w) \in A$, then M accepts input w .

- If $\gamma(w) \notin A$, then M rejects input w .

Note there is no requirement if $w \notin \text{dom}(\gamma)$.

We say that A is **semidecidable** w.r.t. γ if there exists a TM for language recognition that for any $w \in \text{dom}(\gamma)$:

- If $\gamma(w) \in A$, then M accepts input w .
- If $\gamma(w) \notin A$, then M does not accept input w .

For a subset $A \subseteq X$, its **characteristic function** $\chi_A: X \rightarrow \{0,1\}$ is defined as:

$$\chi_A(x) := \begin{cases} 1 & ; x \in A \\ 0 & ; x \notin A \end{cases}$$

Its **partial characteristic function** $\chi_A^p: X \rightarrow \{0,1\}$ is defined as:

$$\chi_A^p(x) := \begin{cases} 1 & ; x \in A \\ \uparrow & ; x \notin A \end{cases}$$

Theorem: A subset $A \subseteq X$ is:

- Decidable iff $\chi_A: X \rightarrow \{0,1\}$ is $(\gamma \rightarrow \gamma_b)$ -computable
- Semidecidable iff $\chi_A^p: X \rightarrow \{0,1\}$ is $(\gamma \rightarrow \gamma_b)$ -computable

