

1) Naj bodo $L, L_1, L_2 \subseteq A^*$ jeziki in $a \in A$. Pokaži:

$$b) a^{-1}L^* = (a^{-1}L)L^*$$

$$L^* = 1 + L + L^2 + \dots$$

$$a^{-1}L^* = \{v \in A^* ; av \in L^* = 1 + L + L^2 + \dots\}$$

$$= a^{-1} \underset{= \emptyset}{1} + a^{-1}L + a^{-1}L^2 + \dots$$

$$= a^{-1}L + a^{-1}L^2 + \dots$$

$$= a^{-1}L(1 + L + L^2 + \dots)$$

$$= (a^{-1}L)L^*$$

2) Naj bodo $L, L_1, L_2, X, Y \subseteq A^*$ jeziki. Pokaži:

$$a) X^{-1}(L_1 + L_2) = X^{-1}L_1 + X^{-1}L_2$$

$$b) (XY)^{-1}L = Y^{-1}(X^{-1}L)$$

$$c) (X^*)^{-1}L = (X^*)^{-1}(X^{-1}L) + L$$

$$\begin{aligned} a) X^{-1}(L_1 + L_2) &= \sum_{u \in X} u^{-1}(L_1 + L_2) \\ &= \sum_{u \in X} (u^{-1}L_1 + u^{-1}L_2) \end{aligned}$$

$$\begin{aligned}
&= \sum_{u \in X} u^{-1} L_1 + \sum_{u \in X} u^{-1} L_2 \\
&= X^{-1} L_1 + X^{-1} L_2
\end{aligned}$$

$$\begin{aligned}
b) (XY)^{-1} L &= \sum_{uv \in XY} (uv)^{-1} L \\
&= \sum_{\substack{uv \in XY \\ u \in X, v \in Y}} v^{-1} (u^{-1} L) \\
&= \sum_{v \in Y} \sum_{u \in X} v^{-1} (u^{-1} L) \\
&= \sum_{v \in Y} v^{-1} \sum_{u \in X} u^{-1} L \\
&= Y^{-1} (X^{-1} L)
\end{aligned}$$

$$\begin{aligned}
c) (X^*)^{-1} L &= \sum_{u \in X^*} u^{-1} L \\
&= 1^{-1} L + \sum_{u \in X^+} u^{-1} L \\
&= L + \sum_{w \in X^*} \sum_{v \in X} (vw)^{-1} L \\
&= L + \sum_{w \in X^*} \sum_{v \in X} w^{-1} (v^{-1} L) \\
&= L + \sum_{w \in X^*} w^{-1} \sum_{v \in X} v^{-1} L \\
&= L + (X^*)^{-1} (X^{-1} L)
\end{aligned}$$

3) Naj bo $\varphi: A^* \rightarrow B^*$ morfizem in $L \subseteq A^*$ racionalni jezik. Pokaži, da je $\varphi(L) \subseteq B^*$ racionalni jezik.

$L \subseteq A^*$ racionalen:

- $\emptyset \in L$
 $\forall a \in A : a \in L$

- Zaprt za končne unije, produkte in zvezdico

$\varphi(\emptyset) = \emptyset$ prazen jezik je racionalen ✓

$\varphi(a \in A) = b \in B^*$ enobesedni jezik tudi racionalen ✓

$$\varphi(L_1 + L_2) = \varphi(L_1) + \varphi(L_2)$$

$$\begin{aligned}\varphi(L_1 + L_2) &= \{ \varphi(u) ; u \in L_1 + L_2 \} \\ &= \{ \varphi(u) ; u \in L_1 \} + \{ \varphi(u) ; u \in L_2 \} \\ &= \varphi(L_1) + \varphi(L_2)\end{aligned}$$

$$\varphi(L_1 L_2) = \varphi(L_1) \varphi(L_2)$$

$$\begin{aligned}\varphi(L_1 L_2) &= \{ \varphi(uv) ; u \in L_1, v \in L_2 \} \\ &= \{ \varphi(u) \varphi(v) ; u \in L_1, v \in L_2 \} \\ &= \{ \varphi(u) ; u \in L_1 \} \cdot \{ \varphi(v) ; v \in L_2 \} \\ &= \varphi(L_1) \varphi(L_2)\end{aligned}$$

$$\varphi(L^*) = (\varphi(L))^* \quad \text{podobno}$$

5) Poenostavi naslednje racionalne izraze:

a) $0^* a b 1^* = *$

$$0^* = \emptyset^* = 1 + 0 + 0^2 + \dots = 1$$

$$1^* = \{1\}^* = 1 + 1 + 1^2 + \dots = 1$$

$$* = 1ab1 = ab$$

$$\begin{aligned} b) ((abb)^*)^* &= (1 + abb + (abb)^2 + \dots)^* \\ &= \left(\sum_{n=0}^{\infty} (abb)^n \right)^* \\ &= 1 + \sum (abb)^n + \left(\sum (abb)^n \right)^2 + \dots \\ &= 1 + \sum (abb)^n + (1 + abb + (abb)^2 + \dots)(1 + abb + (abb)^2 + \dots) + \dots \\ &= \sum_{n=0}^{\infty} (abb)^n \\ &= (abb)^* \end{aligned}$$

$$\text{Opomba: } \left(\sum_{n=0}^{\infty} (abb)^n \right)^k = \sum_{n=0}^{\infty} (abb)^n$$

$$\text{Tudi: } L^{**} = L^*$$

$$c) a^*b + a^*ba^* = a^*b(1 + \underset{=a^*}{a^*}) = a^*ba^*$$

1) Naj bodo $L, L_1, L_2 \subseteq A^*$ jeziki in $a \in A$. Pokaži:

$$a) a^{-1}(L_1 L_2) = \begin{cases} (a^{-1}L_1)L_2 & ; \quad 1 \notin L_1 \\ (a^{-1}L_1)L_2 + a^{-1}L_2 & ; \quad 1 \in L_1 \end{cases}$$

$$a^{-1}L = \{u \in A^* ; au \in L\}$$

$$a^{-1}(L_1 L_2) = \{u \in A^* ; au \in L_1 L_2\} = *$$

i) $1 \notin L_1$:

$$a^{-1}(L_1 L_2) = (a^{-1}L_1)L_2 = \{u \in A^* ; au \in L_1\} L_2$$

$$(a^{-1}L_1)L_2 = \{uv \in A^* ; au \in L_1, v \in L_2, auv \in L_1 L_2\}$$

ii) $1 \in L_1$:

$$a^{-1}(L_1 L_2) = \{uv ; au \in L_1, v \in L_2, u \neq 1\} \\ \cup \{v ; a1 \in L_1 L_2, v \in L_2\}$$

$$= (a^{-1}L_1)L_2 \\ + a^{-1}L_2$$

