

1) Naj bo $x, y \in A^*$. Pokaži, da enakost $xy = yx$ velja natanko tedaj, ko obstaja $z \in A^*$, da je $x = z^n$ in $y = z^m$ za neki neregativni števili n, m .

$$(\Leftarrow) \begin{aligned} x &= z^n \\ y &= z^m \end{aligned}$$

$$xy = z^n z^m = z^{n+m} = z^m z^n = yx$$

$$(\Rightarrow) xy = yx$$

$$\begin{aligned} x &= x_1 \cdots x_n \\ y &= y_1 \cdots y_m \end{aligned}$$

$$\text{BSS (WLOG): } n \geq m$$

$$\begin{aligned} \Rightarrow x_1 &= y_1 \\ x_2 &= y_2 \\ &\vdots \\ x_m &= y_m \end{aligned}$$

$$\Rightarrow x = yw \quad (y \text{ predpona } x)$$

$$xy = \underline{ywy} = yx = \underline{yyw}$$

$$\Rightarrow wy = yw$$

$$w \text{ dolžine } n-m \quad (\text{krajši od } x)$$

Indukcija na $n+m$:

$$n+m = 1:$$

$$n = 1: x = a$$

$$m = 0: y = 1$$

$$\Rightarrow z := a$$

$$\Rightarrow x = z^1 = a^1 = a$$

$$y = z^0 = a^0 = 1$$

$$n+m < M \rightsquigarrow n+m = M:$$

$m = 0 \Rightarrow y$ prazna beseda $\Rightarrow$ Enako kot prej

Denimo: $y \neq 1$ ($m \neq 1$)

$$|w| < |x|$$

Uporabimo induksijsko predpostavko za $|wy| < N$

$\Rightarrow$ Obstaja z , da je $w = z^k$, $y = z^m$

$$\Rightarrow x = wy = z^k z^m = z^{k+m}$$

$$\Rightarrow n = k+m$$

2) Besedi $x, y \in A^*$ sta konjugirani ($x \sim y$), če obstajata $u, v \in A^*$, da je $x = uv$ in $y = vu$. Pokaži, da je $x \sim y$ natanko tedaj, ko obstaja $z \in A^*$, da je $xz = zy$. Sklepaj, da je $\sim$ ekvivalenčna relacija.

$$(\Rightarrow) x \sim y \Rightarrow \exists u, v \in A^*: x = uv, y = vu$$

$$u = u_1 \cdots u_k$$

$$v = v_1 \cdots v_l$$

$$\text{BSS: } k \leq l$$

$$xz = uvz$$

$$zy = zuv$$

$$\Rightarrow z = u$$

$$(\Leftarrow) \exists z \in A^*: xz = zy$$

$$\exists u, v \in A^*: x = uv, y = vu$$

$$1) |z| \leq |x| = |y| \Rightarrow x = zw, y = rz$$

$$z wz = z rz$$

$$\Rightarrow u = z$$

$$v = w$$

$$2) |x| = |y| < |z| \Rightarrow z = xw, z = ry$$

$$xz = zy$$

$$xry = xwy$$

$$r = w$$

$$xz = zy$$

$$\underline{X} \underline{X} \underline{W} = \underline{r} \underline{y}$$

⋮

$$\underline{X} \dots \underline{X} \underline{W}_e = \underline{r}_e \underline{y} \dots \underline{y}$$

$$|\underline{W}_e|, |\underline{r}_e| < |\underline{X}|$$

⇒ Uporabimo presnjo točko

$$\Rightarrow \underline{X} = \underline{U}_e \underline{r}'$$

$$\underline{y} = \underline{r}'' \underline{W}_e$$

$$\underline{r}' = \underline{r}''$$

$$\Rightarrow \underline{u} = \underline{W}_e$$

$$\underline{r}' = \underline{r}''$$

~ ekvivalenčna

$$\underline{X} \sim \underline{Y} \Rightarrow \underline{Y} \sim \underline{X}$$

Očitno

$$\underline{X} \sim \underline{X}$$

Očitno

$$\underline{X} \sim \underline{Y}, \underline{Y} \sim \underline{Z} \Rightarrow \underline{X} \sim \underline{Z}$$

$$\exists w: \underline{X} w = w \underline{Y}$$

$$\exists u: \underline{Y} u = u \underline{Z}$$

$$\text{Iščemo } \underline{r}: \underline{X} \underline{r} = \underline{r} \underline{Z}$$

$$xwu = wyu = wuz$$

$$r = wu$$

