

MODULES

Have R ring. An R -module $\mathcal{M}$ is an abelian group $(\mathcal{M}, +)$ with multiplication $R \times \mathcal{M} \rightarrow \mathcal{M}$, such that
 $(x, m) \mapsto x \cdot m$

$$\begin{aligned} a) \quad x \cdot (m+n) &= x \cdot m + x \cdot n \\ (x+y) \cdot m &= x \cdot m + y \cdot m \end{aligned}$$

... distributivity

$$b) \quad x \cdot (y \cdot m) = (xy) \cdot m$$

... associativity

$$c) \quad 1 \cdot m = m$$

... monicity

For all $x, y \in R$ and all $m, n \in \mathcal{M}$.

A submodule $\mathcal{N} \leq \mathcal{M}$ is a subgroup that is closed under scalar multiplication.

For $I \triangleleft R$, $I \cdot \mathcal{M}$ is the submodule of $\mathcal{M}$, generated by $x \cdot m$, $x \in I$, $m \in \mathcal{M}$:

$$I \cdot \mathcal{M} = \left\{ \sum_{i=1}^n a_i \cdot m_i ; n \in \mathbb{N}, a_i \in I, m_i \in \mathcal{M} \right\}$$

For each $m \in \mathcal{M}$ we associate its annihilator:

$$\text{Ann}_R(m) := \{x \in R; x \cdot m = 0\} \triangleleft R$$

$$\text{Ann}_R(\mathcal{M}) := \{x \in R; x \cdot m = 0 \forall m \in \mathcal{M}\} \triangleleft R$$

$$\mathcal{N} \leq \mathcal{M} \Rightarrow \text{Ann}(\mathcal{M}) \subseteq \text{Ann}(\mathcal{N}) \wedge \text{Ann}(\mathcal{M}) \subseteq \text{Ann}(\mathcal{M}/\mathcal{N})$$

$\mathcal{M}$ is semilocal $\Leftrightarrow$ Have only finitely many max ideals containing $\text{Ann}(\mathcal{M})$

$$\Leftrightarrow R/\text{Ann}(\mathcal{M}) \text{ is semilocal}$$

R ring, $\mathcal{N}, \mathcal{M}$ R -modules

$\alpha: \mathcal{M} \rightarrow \mathcal{N}$ is an R -module homomorphism (an R -linear map) if $\alpha(xm + yn) = x\alpha(m) + y\alpha(n)$ for all $x, y \in R$ and all $m, n \in \mathcal{M}$.

$$\ker \alpha := \alpha^{-1}(0) \leq \mathcal{M}$$

$$\text{im } \alpha := \alpha(\mathcal{M}) \leq \mathcal{N}$$

α is an isomorphism when it is bijective homomorphism. In this case $\alpha^{-1}: \mathcal{N} \rightarrow \mathcal{M}$ is also an isomorphism.

$\text{Hom}_R(\mathcal{M}, \mathcal{N}) = \{\alpha: \mathcal{M} \rightarrow \mathcal{N} \text{ } R\text{-module homomorphism}\}$ is also an R -module:

$$(\alpha + \beta)(m) := \alpha(m) + \beta(m)$$

$$(x \cdot \alpha)(m) := x \cdot \alpha(m) = \alpha(x \cdot m)$$

$$\begin{aligned} \forall \alpha, \beta &\in \text{Hom}_R(\mathcal{M}, \mathcal{N}) \\ \forall m &\in \mathcal{M} \\ \forall x &\in R \end{aligned}$$

$$\mathcal{L} \xrightarrow{\alpha} \mathcal{M}, \mathcal{M} \xrightarrow{\beta} \mathcal{N} \Rightarrow \mathcal{L} \xrightarrow{\beta \circ \alpha} \mathcal{N}$$

$$\mathcal{L} \xrightarrow{\alpha} \mathcal{M}, \mathcal{N} \xrightarrow{\beta} \mathcal{P} \Rightarrow \text{Hom}(\alpha, \beta): \text{Hom}(\mathcal{M}, \mathcal{N}) \rightarrow \text{Hom}(\mathcal{L}, \mathcal{P})$$

$$\gamma \mapsto \beta \gamma \alpha$$

Example: $\mathcal{M}$ R -module $\Rightarrow \text{Hom}_R(R, \mathcal{M}) \cong \mathcal{M}$

$$\varphi \mapsto \varphi(1)$$

$$\text{Note: } \varphi(r) = \varphi(r \cdot 1) = r \cdot \varphi(1)$$

Endomorphisms of $\mathcal{M}$ are homomorphisms from $\mathcal{M}$ to $\mathcal{M}$.

$\text{End}_R(\mathcal{M}) = \text{Hom}_R(\mathcal{M}, \mathcal{M})$ is a noncommutative ring.

For $x \in R$, $\mu_x: \mathcal{M} \rightarrow \mathcal{M}$ is an endomorphism.

$$m \mapsto x \cdot m$$

So, $\mu_R: R \rightarrow \text{End}_{\mathbb{Z}}(\mathcal{M})$ is a ring homomorphism.

$$x \mapsto \mu_x$$

Conversely, if $(\mathcal{N}, +)$ is an abelian group and $\nu: R \rightarrow \text{End}_{\mathbb{Z}}(\mathcal{N})$ is a homomorphism, then $\mathcal{N}$ can be given an R -module structure via:

$$x \cdot n = (\nu(x))(n)$$

$$x \in R, n \in \mathcal{N}$$

Then, $\mu_R = \nu$.

Note: $\ker \nu_R = \text{Ann}(\mathcal{M})$

Module $\mathcal{M}$ is **faithful** if $\nu_R: R \rightarrow \text{End}(\mathcal{M})$ is injective.
Equivalently, $\text{Ann}(\mathcal{M}) = (0)$.

ALGEBRAS

A ring R' is an **R -algebra** if there exists structure map $R \xrightarrow{\varphi} R'$ that is ring homomorphism.

$\mathcal{M}'$ R' -module becomes an R -module by **restriction of scalars**:

$$\underset{\substack{\mathcal{M} \\ R}}{x} \cdot \underset{\substack{\mathcal{M}' \\ R'}}{m} := \underset{\substack{\mathcal{M} \\ R'}}{\varphi(x)} \cdot \underset{\substack{\mathcal{M}' \\ R'}}{m}$$

$$\begin{array}{c} R \xrightarrow{\varphi} R' \xrightarrow{\nu_{R'}} \text{End}_Z(\mathcal{M}) \\ \quad \quad \quad \nearrow \\ \quad \quad \quad \nu_R = \nu_{R'} \circ \varphi \end{array}$$

In particular, R' is an R -module. We have

$$\begin{array}{ccc} (xy)z & = & x(yz) \\ \parallel & & \parallel \\ (\varphi(x)y)z & & \varphi(x)(yz) \end{array} \quad (*)$$

by associativity of R' .

Suppose ring R' is an R -module and the above holds. Then R' is an R -algebra.

Indeed, let's define $\varphi: R \rightarrow R'$
 $x \mapsto x \cdot 1_{R'}$

$$\varphi(x)z = (x \cdot 1_{R'}) \cdot z \stackrel{(*)}{=} x \cdot (1 \cdot z) = xz$$

Scalar multiplication of R on R' is via φ .

