Imamo domnevo $H_0$ in domnevo $H_1$.

Če je model parametriziran s $\theta \in \Theta$, lahko ničelno domnevo $H_0$ določimo s $\theta \in \Theta_0$.

Alternativno domnevo $H_1$ lahko določimo kar z $\Theta_1 = \Theta \setminus \Theta_0$.

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Pri stopnji tveganja $\alpha$ skonstruiramo preizkus, da velja:

$$P_0(H_0 \text{ zavrnjena}) := P(H_0 \text{ zavrnjena} \mid H_0 \text{ drži}) \overset{=}{\leq} \alpha$$

Moč preizkusa definiramo kot:

$$P_1(H_0 \text{ zavrnjena}) := P(H_0 \text{ zavrnjena} \mid H_1 \text{ drži})$$

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Odločitev o zavrnitvi $H_0$ sprejmemo, če preizkusna statistika pade v kritično območje $K_\alpha$.

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| | $H_0$ drži | $H_1$ drži |
|---|---|---|
| $H_0$ zavrnemo | napaka tipa 1 <br> $P(\cdots) \leq \alpha$ | ✓ <br> $P(\cdots) \overset{\text{moč testa}}{=} 1-\beta$ |
| $H_0$ ne zavrnemo | ✓ <br> $P(\cdots) \geq 1-\alpha$ | napaka tipa 2 <br> $P(\cdots) = \beta$ |

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3) Opazujemo $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ NEP. Vemo, da je $\sigma = 5$.
Preizkušamo $H_0: \mu = 100$.

a) Naj bo $H_1: \mu < 100$. Skonstruiraj preizkus stopnje tveganja $\alpha$.

$\hat{\mu} = \overline{X}$ preizkusna statistika

$K_\alpha = (-\infty, c)$

$\overline{X} \in K_\alpha \Rightarrow$ Zavrnemo $H_0$

$P_0(\overline{X} < c) \overset{=}{\leq} \alpha$

$X_i \sim N(100, 5^2)$

$\mathrm{var}(\overline{X}) = \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{var}(X_i) = \frac{25}{n}$

$\Rightarrow \overline{X} \sim N(100, \frac{25}{n})$

$P(\overline{X} < c) = P\left(\frac{\overline{X} - 100}{\frac{5}{\sqrt{n}}} < \frac{c - 100}{\frac{5}{\sqrt{n}}}\right) = F_{N(0,1)}\left(\frac{c-100}{\frac{5}{\sqrt{n}}}\right) = \alpha$

$\frac{c-100}{\frac{5}{\sqrt{n}}} = F^{-1}_{N(0,1)}(\alpha)$

$c - 100 = \frac{5}{\sqrt{n}} F^{-1}_{N(0,1)}(\alpha)$

$c = \frac{5}{\sqrt{n}} F^{-1}_{N(0,1)}(\alpha) + 100$

b) Naj bo $H_1: \mu \neq 100$. Skonstruiraj preizkus stopnje tveganja $\alpha$.


$K_\alpha = (-\infty, \mu_0 - d) \cup (\mu_0 + d, \infty)$

$|\overline{X}-\mu_0| > d \Rightarrow$ Zavrnemo $H_0$

$$P(|\overline{X}-\mu_0| > d) \overset{=}{\leq} \alpha$$

$$2P(\overline{X}-\mu_0 > d) \leq \alpha$$

$$2P\left(\frac{\overline{X}-\mu_0}{\frac{5}{\sqrt{n}}} > \frac{d}{\frac{5}{\sqrt{n}}}\right) \leq \alpha$$

$$1 - F_{N(0,1)}\left(\frac{d}{\frac{5}{\sqrt{n}}}\right) \leq \frac{\alpha}{2}$$

$$d \cdot \frac{\sqrt{n}}{5} \geq F^{-1}_{N(0,1)}\left(1-\frac{\alpha}{2}\right)$$

$$d \geq \frac{5}{\sqrt{n}} F^{-1}_{N(0,1)}\left(1-\frac{\alpha}{2}\right)$$

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4) Imamo podatke $X_1, \dots, X_5 \sim N(1, \sigma^2)$ NEP o finančni naložbi. Z verjetnostjo vsaj $1-\alpha = 0{,}95$ se želimo izogniti investiranju, ko je $\sigma^2 \geq \sigma_0^2$ za nek znan $\sigma_0^2 > 0$. Sestavi algoritem za sprejetje odločitve.

$$\hat{\sigma}^2 = \frac{1}{5}\sum_{i=1}^{5}(X_i - 1)^2$$

$$T_{\sigma^2} := \frac{\hat{\sigma}^2 \cdot 5}{\sigma^2} \sim \chi^2(5)$$, če je $var(X_i) = \sigma^2$

Odločitev o investiranju bo najbolj tvegana, ko je $\sigma^2 = \sigma_0^2$:

$P_{\sigma^2}(T_{\sigma_0^2} < c)$ padajoča v $\sigma^2$

$\Rightarrow P_{\sigma^2}(T_{\sigma_0^2} < c)$ je na $[\sigma_0^2, \infty)$ maksimalna pri $\sigma = \sigma_0$

Investiramo, če bo $T_{\sigma^2} < c$.

$$\iff \sum (X_i - 1)^2 < \sigma_0^2 \cdot c$$

Ko je $\sigma^2 > \sigma_0^2$, bomo investirali z verjetnostjo največ $\alpha$.

$$P_{\sigma_0^2}(T_{\sigma_0^2} < c) = \alpha$$

$$F_{\chi^2(5)}(c) = \alpha$$

$$c = F^{-1}_{\chi^2(5)}(\alpha)$$

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Preizkus na podlagi razmerja verjetij:

Imamo:

$H_0$ s $\Theta_0$
$H_1$ s $\Theta_1 = \Theta \setminus \Theta_0$

Preizkus na podlagi razmerja verjetij temelji na statistiki:

$$\Lambda = \frac{\sup_{\theta \in \Theta} L(\theta | X)}{\sup_{\theta \in \Theta_0} L(\theta | X)}$$

Oziroma:

$$\Lambda' = \frac{\sup_{\theta \in \Theta_1} L(\theta | X)}{\sup_{\theta \in \Theta_0} L(\theta | X)}$$

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Wilksov izrek:

$$2 \ln \Lambda \xrightarrow[n \to \infty]{d} \chi^2(\dim \Theta - \dim \Theta_0)$$

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5) $X$ ... število klicev v Kopru na dan $\sim Po(\lambda_X)$
$Y$ ... število klicev v Celju na dan $\sim Po(\lambda_Y)$

$n = 100$
$\overline{X} = 20$
$\overline{Y} = 22$

$\alpha = 0{,}05$ stopnja

$H_0 : \lambda_X = \lambda_Y$
$H_1 : \lambda_X \neq \lambda_Y$

$\Theta = [0, \infty)^2$
$\Theta_0 = [0, \infty)$

$P(X=k) = \frac{\lambda_X^k}{k!} e^{-\lambda_X}$
$P(Y=k) = \frac{\lambda_Y^k}{k!} e^{-\lambda_X}$

$$L(\lambda_X, \lambda_Y | X, Y) = \prod_{i=1}^{n} \frac{\lambda_X^{x_i}}{x_i!} e^{\lambda_X} \cdot \frac{\lambda_Y^{y_i}}{y_i!} e^{\lambda_Y}$$

$$\ell(\lambda_X, \lambda_Y | X, Y) = \ln L = \sum_{i=1}^{n} \left(-\lambda_X + \ln \frac{\lambda_X^{x_i}}{x_i!} - \lambda_Y + \ln \frac{\lambda_Y^{y_i}}{y_i!}\right)$$
$$= -n\lambda_X - n\lambda_Y + \sum_{i=1}^{n} x_i \ln \lambda_X - \ln x_i! + y_i \ln \lambda_Y - \ln y_i!$$

i) $H_0 : \lambda_X = \lambda_Y = \lambda$

$$\ell = -2n\lambda + \sum_{i=1}^{n} (x_i + y_i) \ln \lambda$$

$$\frac{\partial \ell}{\partial \lambda} = -2n\lambda + \sum_{i=1}^{n} \frac{x_i + y_i}{\lambda} = 0$$

$$\Rightarrow 2n = \frac{1}{\lambda} \sum_{i=1}^{n} (x_i + y_i)$$

$$\Rightarrow \lambda = \frac{1}{2n} \sum_{i=1}^{n} (x_i + y_i) = \frac{1}{2}(\overline{X} + \overline{Y})$$

To je maksimum, ker je drugi odvod manjši od 0.

ii) $\frac{\partial \ell}{\partial \lambda_x}, \frac{\partial \ell}{\partial \lambda_y} = \dots$

$\vdots$

$$\Rightarrow \lambda_x = \overline{X}$$
$$\lambda_y = \overline{Y}$$

$$\Lambda = \frac{\prod \frac{\overline{X}^{x_i}}{x_i!} \cdot e^{-\overline{X}} \cdot \frac{\overline{Y}^{y_i}}{y_i!} \cdot e^{-\overline{Y}}}{\prod \frac{\left(\frac{1}{2}(x_i+y_i)\right)^{x_i+y_i}}{y_i!\, x_i!} \cdot e^{-(\overline{X}+\overline{Y})}} = \dots = \frac{\overline{X}^{n\overline{X}}\, \overline{Y}^{n\overline{Y}}}{\left(\frac{1}{2}(\overline{X}+\overline{Y})\right)^{n\overline{X}+n\overline{Y}}}$$

$H_0$ zavrnemo, če je $\Lambda > c$.

Wilksov izrek: $2\ln\Lambda \xrightarrow[n\to\infty]{d} \chi^2(1)$

$$P(\overbrace{2\ln\Lambda}^{\sim\chi^2(1)} > d) = \alpha$$

$$1 - P(2\ln\Lambda < d) = \alpha$$

$$1-\alpha = P(2\ln\Lambda < d)$$

$$\underline{d = F^{-1}_{\chi^2(1)}(1-\alpha)} \approx 3{,}84$$

V našem primeru dobimo $2\ln\Lambda \approx 9{,}5$.

Torej $H_0$ zavrnemo.