3) Naj bodo $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ NEP. Skonstruiraj intervale zaupanja stopnje zaupanja $1-\alpha$ / stopnje napake α za:

a) μ, če σ poznamo:

$$P(\mu \in (\overline{X}-y, \overline{X}+y)) \geq 1-\alpha$$

$$P(\overline{X}-y < \mu < \overline{X}+y) \geq 1-\alpha$$

$$P(\overline{X}-\mu-y < 0 < \overline{X}-\mu+y) \geq 1-\alpha$$

$$P(|\overline{X}-\mu| < y) \geq 1-\alpha$$

$$P(|\overline{X}-\mu| > y) \leq \alpha$$

$$2P(X-\mu > y) \leq \alpha$$

$$2 \cdot \left(1-F_{N(0,1)}\left(\frac{y}{\sigma/\sqrt{n}}\right)\right) \leq \alpha$$

$$F_{N(0,1)}\left(\frac{y}{\sigma/\sqrt{n}}\right) \geq 1-\frac{\alpha}{2}$$

$$\frac{y}{\sigma/\sqrt{n}} \geq F^{-1}_{N(0,1)}\left(1-\frac{\alpha}{2}\right)$$

$$y \geq \frac{\sigma}{\sqrt{n}} F^{-1}_{N(0,1)}\left(1-\frac{\alpha}{2}\right)$$

Hitrejši način:

$$Var(\overline{X}) = \frac{\sigma}{\sqrt{n}}$$

$$T = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$P\left(-F^{-1}_{N(0,1)}\left(1-\frac{\alpha}{2}\right) \leq T \leq F^{-1}_{N(0,1)}\left(1-\frac{\alpha}{2}\right)\right) = 1-\alpha$$

$$-F^{-1}\left(1-\frac{\alpha}{2}\right) \leq \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \leq F^{-1}\left(1-\frac{\alpha}{2}\right)$$

$$\overline{X} - \frac{\sigma}{\sqrt{n}} F^{-1}\left(1-\frac{\alpha}{2}\right) \leq \mu \leq \overline{X} + \frac{\sigma}{\sqrt{n}} F^{-1}\left(1-\frac{\alpha}{2}\right)$$

b) μ, če σ ne poznamo:

$$P(|\overline{X}-\mu| < \sigma) \geq 1-\alpha$$

$$T = \frac{\overline{X}-\mu}{\frac{\hat{\sigma}_+}{\sqrt{n}}} \sim \text{Student}(n-1)$$

$$P\left(-F_S^{-1}\left(1-\frac{\alpha}{2}\right) < T < F_S^{-1}\left(1-\frac{\alpha}{2}\right)\right) = 1-\alpha$$

$$-F_S^{-1}\left(1-\frac{\alpha}{2}\right) < \frac{\overline{X}-\mu}{\frac{\hat{\sigma}_+}{\sqrt{n}}} < F_S^{-1}\left(1-\frac{\alpha}{2}\right)$$

$$-\frac{\hat{\sigma}_+}{\sqrt{n}} F_S^{-1}\left(1-\frac{\alpha}{2}\right) < \overline{X}-\mu < \frac{\hat{\sigma}_+}{\sqrt{n}} F_S^{-1}\left(1-\frac{\alpha}{2}\right)$$

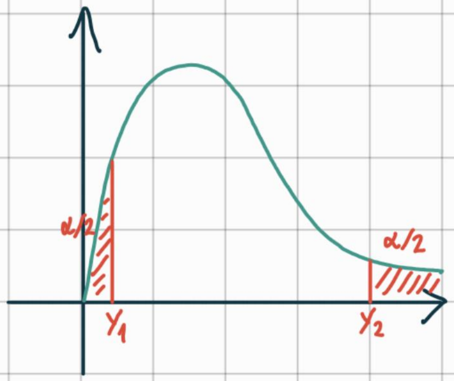
$$\frac{\hat{\sigma}_+}{\sqrt{n}} F_S^{-1}\left(1-\frac{\alpha}{2}\right) > \mu-\overline{X} > -\frac{\hat{\sigma}_+}{\sqrt{n}} F_S^{-1}\left(1-\frac{\alpha}{2}\right)$$

$$\overline{X} - \frac{\hat{\sigma}_+}{\sqrt{n}} F_S^{-1}\left(1-\frac{\alpha}{2}\right) < \mu < \overline{X} + \frac{\hat{\sigma}_+}{\sqrt{n}} F_S^{-1}\left(1-\frac{\alpha}{2}\right)$$

c) za σ, če μ ne poznamo:

$$\frac{\hat{\sigma}_+^2}{\sigma^2} = \frac{\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\overline{X})^2}{n-1} \sim \chi^2(n-1)$$

$$P\left(F^{-1}_{\chi^2}\left(\frac{\alpha}{2}\right) \leq \frac{\hat{\sigma}_+^2}{\sigma^2} \leq F^{-1}_{\chi^2}\left(1-\frac{\alpha}{2}\right)\right) = 1-\alpha$$



$y_1 = F^{-1}_{\chi^2}\left(\frac{\alpha}{2}\right)$
$y_2 = F^{-1}_{\chi^2}\left(1-\frac{\alpha}{2}\right)$

$$P\left(\frac{1}{F^{-1}_{\chi^2(n)}\left(\frac{\alpha}{2}\right)} \geq \frac{\sigma^2}{\hat{\sigma}_+^2} \geq \frac{1}{F^{-1}_{\chi^2(n)}\left(1-\frac{\alpha}{2}\right)}\right) = P\left(\frac{\hat{\sigma}_+}{\sqrt{F^{-1}_{\chi^2(n-1)}\left(1-\frac{\alpha}{2}\right)}} \leq \sigma \leq \frac{\hat{\sigma}_+}{\sqrt{F^{-1}_{\chi^2(n-1)}\left(\frac{\alpha}{2}\right)}}\right) = 1-\alpha$$

d) za σ, če μ poznamo:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$$

$$T = \frac{1}{\sigma^2}\hat{\sigma}^2 \sim \chi^2(n)$$

$$P\left(F^{-1}_{\chi^2(n)}\left(\frac{\alpha}{2}\right) \leq \frac{\hat{\sigma}^2}{\sigma^2} \leq F^{-1}_{\chi^2(n)}\left(1-\frac{\alpha}{2}\right)\right) = 1-\alpha$$

$$P\left(\frac{\hat{\sigma}^2}{F^{-1}_{\chi^2(n)}\left(1-\frac{\alpha}{2}\right)} \leq \sigma^2 \leq \frac{\hat{\sigma}^2}{F^{-1}_{\chi^2(n)}\left(\frac{\alpha}{2}\right)}\right) = 1-\alpha$$

29.3.

1) θ ... verjetnost neuspeha Bernoullijevega poskusa
β ... stopnja zaupanja

Skonstruiraj interval zaupanja za θ, ki temelji na n poskusih.

Če se kateri poskus ponesreči, vzamemo interval zaupanja $[0,1]$, sicer pa vzamemo interval $[0, \theta^*)$.

Poišči najmanjši možni θ^*.

$P(\theta \in IZ) \geq \beta$

$\theta < \theta^*$: Ok

$\theta \geq \theta^*$: Vsaj en poskus že mora ponesrečiti, da bo θ v IZ

Iščemo θ^*, da bo $P(\text{vsaj en ponesrečen}) \geq \beta$.

$$P(\text{vsaj en ponesrečen}) = 1 - P(\text{ni ponesrečenih}) = 1-(1-\theta)^n \geq \beta$$

$$1-\beta \geq (1-\theta)^n$$

$$\sqrt[n]{1-\beta} \geq 1-\theta$$

$$\theta \geq 1-\sqrt[n]{1-\beta}$$

$$\Rightarrow \theta^* = 1-\sqrt[n]{1-\beta} \quad \text{(najmanjša možna)}$$

Naj bo $\hat{\theta}$ cenilka po metodi največjega verjetja v dovolj lepem modelu.

Pri dovolj velikih n je $\hat{\theta}$ porazdeljena približno $N(\theta, \frac{1}{FI(\theta)})$.

To nam da asimptotični interval zaupanja stopnje zaupanja $1-\alpha$ oblike $\left[\hat{\theta} - \frac{F^{-1}_{N(0,1)}(1-\frac{\alpha}{2})}{\sqrt{FI(\hat{\theta})}}, \hat{\theta} + \frac{F^{-1}_{N(0,1)}(1-\frac{\alpha}{2})}{\sqrt{FI(\hat{\theta})}}\right]$, kjer velja $FI(\theta) = n FI_1(\theta)$.

---

2) Naj bodo $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ neodvisni.

Določi interval zaupanja za σ^2.

Vemo, da je interval zaupanja za σ enak:

$$\left[\hat{\sigma}_+ \sqrt{\frac{n-1}{F^{-1}_{\chi^2(n-1)}(1-\frac{\alpha}{2})}}, \hat{\sigma}_+ \sqrt{\frac{n-1}{F^{-1}_{\chi^2(n-1)}(\frac{\alpha}{2})}}\right]$$

$$\hat{\sigma}_+^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Delajmo se, da tega ne vemo. Zanima nas asimptotski interval.

$$L_1(\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\Rightarrow \ell(\mu,\sigma) = \sum_{i=1}^{n}\left(\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$= \sum_{i=1}^{n}\left(-\ln\sigma - \ln\sqrt{2\pi} - \frac{(x-\mu)^2}{2\sigma^2}\right)$$

Odvajamo in poiščemo ekstreme ...

Dobimo:

$$\hat{\mu} = \overline{X}$$

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$$

Potrebujemo še druge odvode:

$$\frac{\partial^2}{\partial\mu^2}\ell_1(\mu,\sigma) = -\frac{1}{\sigma^2}$$

$$\frac{\partial^2}{\partial\mu\partial\sigma}\ell_1(\mu,\sigma) = -\frac{2(x_1-\mu)}{\sigma^3}$$

$$\frac{\partial^2}{\partial\sigma^2}\ell_1(\mu,\sigma) = \frac{1}{\sigma^2} - \frac{3(x_1-\mu)^2}{\sigma^4}$$

$$\Rightarrow FI_1(\mu,\sigma) = -E\begin{bmatrix}\frac{\partial^2}{\partial\mu^2}\ell_1 & \frac{\partial^2}{\partial\mu\partial\sigma}\ell_1 \\ \frac{\partial^2}{\partial\mu\partial\sigma}\ell_1 & \frac{\partial^2}{\partial\sigma^2}\ell_1\end{bmatrix} = -\begin{bmatrix}-\frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2}-\frac{3}{\sigma^2}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2}\end{bmatrix}$$

$$\theta = \begin{bmatrix}\mu \\ \sigma^2\end{bmatrix}$$

$$\sigma^2 = h(\theta)$$

$$\nabla h^T \cdot (nFI_1)^{-1} \cdot \nabla h = [0 \ 1](nFI_1)^{-1}\begin{bmatrix}0\\1\end{bmatrix} = \frac{\hat{\sigma}^2}{2n} \sim \mathrm{var}(\hat{\sigma}^2)$$

Asimptotski interval zaupanja:

$$\left[\hat{\sigma} - \frac{F^{-1}_{N(0,1)}(1-\frac{\alpha}{2})\,\hat{\sigma}}{\sqrt{2n}},\ \hat{\sigma} + \frac{F^{-1}_{N(0,1)}(1-\frac{\alpha}{2})\,\hat{\sigma}}{\sqrt{2n}}\right]$$