Standardna p-razsežna normalna porazdelitev je porazdelitev vektorja $Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_p \end{bmatrix}$, kjer so $Z_i \sim N(0,1)$ NEP.

Splošna n-razsežna normalna porazdelitev je vsaka porazdelitev slučajnega vektorja $X = AZ + \mu$, kjer je Z porazdeljen standardno normalno, $A \in \mathbb{R}^{n \times p}$ in $\mu \in \mathbb{R}^n$.

Ta porazdelitev je enolično določena s pričakovano vrednostjo $\mu \in \mathbb{R}^n$ in kovariančno matriko Σ.

Oznaka: $X \sim N(\mu, \Sigma)$

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5) Naj bo $V := \begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}\right)$. Določi $P(X>0, Y>0)$.

Obstajata A in μ, da je $X = AZ + \mu$, kjer je $Z \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, I_2\right)$.

$$E(V) = E(AZ + \mu) = AE(Z) + \mu = \mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Sigma = Var(V) = Var(AZ + \mu) = A\,Var(Z)A^T = AA^T$$

Razcep Choleskega:

$$a_{jj} = \sqrt{\Sigma_{jj} - \sum_{k=1}^{j-1} a_{jk}^2}$$

$i \geq j: \quad a_{ij} = \frac{\Sigma_{ij} - \sum_{k=1}^{j-1} a_{ik} a_{jk}}{a_{jj}}$

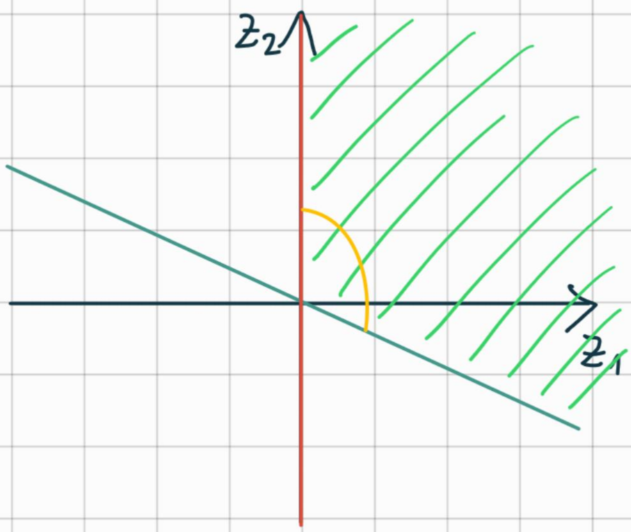
$\Rightarrow A = \begin{bmatrix} 2\sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \end{bmatrix}$

$\Rightarrow V = AZ$

$\Rightarrow X = 2\sqrt{2} Z_1$
$\quad\; Y = \frac{\sqrt{2}}{2} Z_1 + \frac{3\sqrt{2}}{2} Z_2$

$\Rightarrow P(X>0, Y>0) = P(2\sqrt{2} Z_1 > 0, \frac{\sqrt{2}}{2} Z_1 + \frac{3\sqrt{2}}{2} Z_2 > 0)$

$= P(Z_1 > 0, Z_2 > -\frac{1}{3} Z_1) = *$



Std. norm. je simetrična okoli $\vec{0}$.

$\Rightarrow * = \frac{\pi/2 + \operatorname{arctg}\frac{1}{3}}{2\pi}$

15.5.

Naj bo $Z \sim N(\vec{0}, I_n)$. Potem je $\|Z\|^2 = Z^2 + \dots + Z^2 \sim \chi^2(n) = \Gamma(\frac{n}{2}, \frac{1}{2})$.

Naj bo H ortogonalni projektor ranga p $(H = H^T = H^2)$. Potem je $\|HZ\|^2 \sim \chi^2(p)$.

Naj bo še $K \sim \Gamma(\frac{p}{2}, \frac{1}{p}) = \chi^2(p)$, kjer sta Z in K neodvisni. Potem je $T := \frac{Z}{\sqrt{\frac{1}{p}K}} \sim Student(p)$.

Če so $X_i \sim N(\mu, \sigma^2)$ NEP, potem je $\frac{\overline{X}-\mu}{\sigma}\sqrt{n} \sim N(0,1)$.

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1) $X_i \sim N(\mu, \sigma^2)$ NEP

$$\hat{\sigma}_+^2 = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2}$$

$$\frac{\overline{X}-\mu}{\hat{\sigma}_+}\sqrt{n} \sim ?$$

$$\frac{\overline{X}-\mu}{\hat{\sigma}_+}\sqrt{n} = \frac{\frac{\overline{X}-\mu}{\sigma}\sqrt{n}}{\sqrt{\frac{\hat{\sigma}_+^2}{\sigma^2}}} \sim N(0,1)$$

$$\sqrt{\frac{\hat{\sigma}_+^2}{\sigma^2}} = \sqrt{\frac{\frac{1}{\sigma^2}\sum(X_i-\overline{X})^2}{n-1}}$$

$$\sum_{i=1}^{n}(Z_i-\overline{Z})^2 = \|HZ\|^2$$

$$H = I - \frac{1}{n}1_n 1_n^T \quad , \quad \text{rang} H = n-1$$

$$\Rightarrow \frac{1}{\sigma^2}\sum(X-\overline{X})^2 \sim \chi^2(n-1)$$

Dovolj je pokazati, da sta neodvisna $\overline{X}$ in HX.

Ker sta $\overline{X}$ in HX normalna slučajna vektorja, neodvisnost sledi iz nekoreliranosti, zato moramo pokazati slednjo.

$$\overline{X} = AX = \frac{1}{n}1_n^T X$$

$$\begin{aligned} Cov(\overline{X}, HX) &= Cov(\tfrac{1}{n}1_n^T X, HX) \\ &= \tfrac{1}{n}1_n^T\, Cov(X,X) H^T \end{aligned}$$

$$= \frac{1}{n} 1_n^T \sigma^2 I_n H$$

$$= \frac{1}{n} 1_n^T \sigma^2 (I_n - \frac{1}{n} 1_n 1_n^T)$$

$$= \frac{\sigma^2}{n} 1_n^T - \frac{\sigma^2}{n} 1_n^T$$

$$= 0$$

$$\Rightarrow \frac{\bar{X} - \mu}{\hat{\sigma}_+} \sqrt{n} \sim \text{Student}(n-1) = t(n-1)$$

---

2) $Z \sim N(\vec{0}, I_n)$

a) H parcialna izometrija $\mathbb{R}^n \to \mathbb{R}^m$:

$\exists V \leq \mathbb{R}^n$ podprostor: $H|_V$ izometrija, $H|_{V^\perp} \equiv 0$

($\Leftrightarrow H^T H$ projektor)

$\|HZ\|^2 \sim ?$

$p = \operatorname{rang} H$

$Q \in O(\mathbb{R}^{n \times n})$ preslika V na $\mathbb{R}^p \times \{0\}^{n-p}$

$$F : \mathbb{R}^n \to \mathbb{R}^m$$
$$\underset{\substack{\cap \\ \mathbb{R}^n}}{(x}, \underset{\substack{\cap \\ \mathbb{R}^{n-p}}}{y)} \mapsto (x, 0)$$

$P \in O(\mathbb{R}^{m \times m})$ preslika $\mathbb{R}^p \times \{0\}^{m-p}$ v $\operatorname{Im} H$

$\Rightarrow H = PFQ$

$$\|HZ\|^2 = \|PFQZ\|^2 = \|FQZ\|^2 = *$$

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix} = QZ \sim N(0, I_n)$$

$$\begin{bmatrix} W_1 \\ \vdots \\ W_p \\ 0 \end{bmatrix} = FQZ \sim N(0, I_p)$$

$$* = \sum_{i=1}^{p} W_i^2 \sim \chi^2(p)$$

b) H ortogonalni projektor

$$Z^T H Z \sim ?$$

$$\begin{aligned} Z^T H Z &= Z^T H \cdot H Z \\ &= Z H^T \cdot H Z \\ &= (HZ)^T \cdot HZ \\ &= \langle HZ, HZ \rangle \\ &= \|HZ\|^2 \sim \chi^2(p), \quad \text{rang}\, H = p \end{aligned}$$

c) $X \sim N(\vec{0}, \Sigma)$, Σ ortogonalni projektor

$$\|X\|^2 \sim ?$$

$$\exists A : AZ = X$$

$$\begin{aligned} \mathrm{Var}(AZ) &= \mathrm{Var}X = \Sigma \\ &\| \\ A\,\mathrm{Var}Z\,A^T & \\ &\| \\ AA^T &= \Sigma \end{aligned}$$

$$A = \Sigma$$

$$X = \Sigma Z$$

$$\|X\|^2 = \|\Sigma Z\|^2 \sim \chi^2(p), \quad p = \mathrm{rang}\,\Sigma$$