(Logično nadaljevanje 4. poglavja)

METODA EMPIRIČNE PORAZDELITVE

Opazujemo vrednosti $X_1, X_2, \dots, X_n$, ki so enako porazdeljene: Predstavljajo vzorec neke porazdelitve. Preučevana količina Y naj se izraža kot karakteristika te porazdelitve:

$$Y = g(\theta) = \mathrm{Char}_\theta(X_1) = \mathrm{Char}(\text{porazdelitev za } X_1)$$

parameter porazdelitve

Tipična primera:

1) $\mathrm{Char}_\theta(X_1) = E_\theta(X_1)$

2) $\mathrm{Char}_\theta(X_1) = \mathrm{var}_\theta(X_1)$

Cenilka za Y je potem $\hat{Y} = \mathrm{Char}(\mathrm{Emp}(X_1, X_2, \dots, X_n))$:

1) $Y = \mu = E_\theta(X_1)$: $\quad \hat{Y} = \hat{\mu} = \frac{X_1 + X_2 + \dots + X_n}{n} = \overline{X}$

2) $Y = \sigma^2 = \mathrm{var}_\theta(X_1)$: $\quad \hat{Y} = \hat{\sigma}^2 = \frac{(X_1 - \overline{X})^2 + (X_2 - \overline{X})^2 + \dots + (X_n - \overline{X})^2}{n}$

Kako dobri sta ti dve cenilki?

1) Obravnavajmo cenilko za $E(X_1)$.

$\overline{X}$ je nepristranska cenilka za $E(X_1)$.

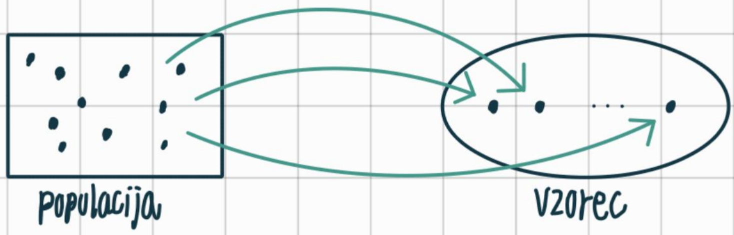
Koliko pa je standardna napaka?

i) Če so $X_1, X_2, \dots, X_n$ nekorelirane, je $SE = \sqrt{var_\theta(\bar{X})} = \frac{\sigma}{\sqrt{n}}$

ii) Enostavno slučajno vzorčenje:

Na populaciji z N enotami naj bo definirana statistična spremenljivka, torej funkcija $\{1,2,\dots,N\} \to \mathbb{R}$, $1 \mapsto x_1$, $2 \mapsto x_2, \dots,$ $N \mapsto x_N$.

Iz populacije vzamemo vzorec, ki naj ga sestavljajo enote z indeksi $K_1, K_2, \dots, K_n \in \{1,2,\dots,N\}$, ki so izbrani naključno.



O enostavnem slučajnemu vzorčenju govorimo, če so vsi možni vzorci brez ponavljanja enako verjetni, torej če je slučajni vektor $(K_1, K_2, \dots, K_n)$ porazdeljen enakomerno na množici $\{(k_1, k_2, \dots, k_n) \in \{1,2,\dots,N\}^N ;\ k_1, k_2, \dots, k_n \text{ vsi različni}\}$.

$X_i := x_{K_i}$, $i = 1,2,\dots,n$

Kje je vir slučaja? V indeksih k.

Primer: $N = 10$

k:	1	2	3	4	5	6	7	8	9	10
x_k:	28	32	42	17	6	87	93	53	41	37

V vzorec velikosti 3 vzamemo 6., 3. in 9. enoto:

$$K_1 = 6 \quad \Rightarrow \quad X_1 = 87$$
$$K_2 = 3 \qquad\qquad X_2 = 42$$
$$K_3 = 9 \qquad\qquad X_3 = 41$$

$$\Theta := \mathbb{R}^N$$

Vzamemo $(\Omega, \mathcal{F}, P)$

$$K_1, K_2, \dots, K_n : \Omega \to \{1, 2, \dots, N\}$$

Izberemo $\theta := (x_1, x_2, \dots, x_N) \in \Theta$

$$P_\theta := P$$

$$X_i(\omega, \theta) = x_{K_i}(\omega)$$

$\mathrm{var}(\overline{X}) = ?$

$$X_1, X_2, \dots, X_N \sim \mathrm{Emp}(x_1, x_2, \dots, x_N)$$

$$\mu := E(X_1) = \frac{x_1 + x_2 + \dots + x_N}{N}$$

$$\sigma^2 := \mathrm{var}(X_1) = \frac{(x_1-\mu)^2 + (x_2-\mu)^2 + \dots + (x_N-\mu)^2}{N}$$

$$SE^2 = \mathrm{var}(\overline{X}) = \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i,j} \mathrm{cov}(X_i, X_j)$$

- $i = j$:

$$\mathrm{cov}(X_i, X_j) = \sigma$$

- $i \neq j$:

$$E(X_i X_j) = E(x_{K_i} x_{K_j})$$

$$= \sum_{k,l} x_k x_l P(K_i = k, K_j = l) = *$$

$$P(K_i = k, K_j = l) = \begin{cases} \frac{1}{N(N-1)} & ; k \neq l \\ 0 & ; k = l \end{cases}$$

$$* = \sum_{k \neq l} x_k x_l \cdot \frac{1}{N(N-1)}$$

$$= \frac{1}{N(N-1)} \sum_{k,l} x_k x_l - \frac{1}{N(N-1)} \sum_k x_k^2$$

$$= \frac{1}{N(N-1)} \left(\sum_k x_k\right)^2 - \frac{1}{N(N-1)} \sum_k x_k^2$$

$$= \frac{N}{N-1} \left(\frac{1}{N} \sum_k x_k\right)^2 - \frac{1}{N-1} \frac{1}{N} \sum_k x_k^2$$

$$= \frac{N}{N-1} E(X_1)^2 - \frac{1}{N-1} E(X_1^2)$$

$$= -\frac{1}{N+1} \sigma^2 + N\mu$$

$$\Rightarrow \operatorname{cov}(X_i, X_j) = -\frac{\sigma^2}{N-1} \qquad \text{za } i \neq j$$

$$\Rightarrow SE^2 = \frac{1}{n^2}\left(n\sigma^2 - n(n-1) \cdot \frac{1}{N-1} \sigma^2\right)$$

$$= \frac{\sigma^2}{n}\left(1 - \frac{n-1}{N-1}\right)$$ korelacijski člen

$$= \frac{\sigma^2}{n} \cdot \frac{N-n}{N-1}$$ korelacijski faktor

Korelacijski člen oziroma faktor je popravek za končne populacije.

Opomba: Tudi pri enostavnem slučajnem vzorčenju se izkaže, da je $\bar{\mu}$ NNLC za μ.

2) Obravnavajmo cenilko za $\operatorname{var}(X_1)$.

$$Y = var(X_1) = \sigma^2$$

$$\hat{\sigma}^2 = \frac{(x_1-\overline{x})^2+(x_2-\overline{x})^2+\ldots+(x_n-\overline{x})^2}{n}$$

Je ta cenilka nepristranska? Koliko je torej $E(\hat{\sigma}^2)$?

$$\hat{\sigma} = \frac{1}{n} \left\| \begin{bmatrix} x_1-\overline{x} \\ x_2-\overline{x} \\ \vdots \\ x_n-\overline{x} \end{bmatrix} \right\|^2$$

$$\underline{X} := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \underline{1}_n := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\underline{\underline{H}} := \underline{\underline{I}}_n - \frac{1}{n} \underline{1}_n \underline{1}_n^T$$

$$\underline{X} = \frac{1}{n} \cdot \underline{1}_n^T \underline{X}$$

Kaj je zaloga vrednosti matrike $\underline{1}_n \underline{1}_n^T$?

$\frac{1}{n} \underline{1}_n (\underline{1}_n^T \underline{x})$ ima vse komponente enake.

$$\underline{x} = \alpha \underline{1}_n : \quad \frac{1}{n} \underline{1}_n \underline{1}_n^T \underline{X} = \alpha \frac{1}{n} \underline{1}_n \underbrace{(\underline{1}_n^T \underline{1}_n)}_{n} = \alpha \underline{1}_n = x$$

$\underline{1}_n \underline{1}_n^T$ je torej projektor na podprostor vektorjev, ki imajo vse komponente enake.

$$rang\left(\frac{1}{n} \underline{1}_n \underline{1}_n^T\right) = 1$$

$$\left(\frac{1}{n} \underline{1}_n \underline{1}_n\right)^T = \frac{1}{n} \underline{1}_n \underline{1}_n^T$$

Gre torej za ortogonalni projektor.

Torej je tudi $\underline{\underline{H}}$ ortogonalni projektor ranga $n-1$.

$$\hat{\sigma}^2 = \frac{1}{n}\underline{X}^T \underline{\underline{H}}\,\underline{X}$$

$X_1, X_2, \dots, X_n$ enako porazdeljene

$E(X_1) = \mu$

$var(X_1) = \sigma^2, \quad \sigma > 0$

Cenilka za μ po metodi empirične porazdelitve:

$$\bar{\mu} = \bar{X} = \frac{1}{n}\underline{1}^T \underline{X}$$

$$\underline{X} := \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}, \quad \underline{1}_n := \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Cenilka za σ^2 po metodi empirične porazdelitve:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n}\underline{X}^T \underline{\underline{H}}\,\underline{X}$$

$\underline{\underline{H}} := I_n - \frac{1}{n}\underline{1}_n \underline{1}_n^T$ ... ortogonalni projektor ranga $n-1$

$$\underline{\underline{H}}^2 = \underline{\underline{H}}^T = \underline{\underline{H}}$$

Vemo: $Var(\underline{X}) = E(\underline{X}\,\underline{X}^T) - E(\underline{X})E(\underline{X})^T$

Trik 1: $\hat{\sigma}^2 = \frac{1}{n} sl(\underline{X}^T \underline{\underline{H}}\,\underline{X}) = \frac{1}{n} sl(\underline{\underline{H}}\,\underline{X}\,\underline{X}^T)$

Trik 2: $Y_i := X_i - \mu$

$$\underline{Y} = \underline{X} - \underline{\mu}\,\underline{1}_n$$

$$\bar{Y} = \frac{Y_1 + Y_2 + \dots + Y_n}{n} = \bar{X} - \mu$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i-\overline{Y})^2 = \frac{1}{n}\underline{Y}^T \underline{\underline{H}}\,\underline{Y} = \frac{1}{n}\,\mathrm{sl}(\underline{\underline{H}}\,\underline{Y}\,\underline{Y}^T)$$

$$E(\hat{\sigma}^2) = \frac{1}{n}\,\mathrm{sl}(\underline{\underline{H}}\,E(\underline{Y}\,\underline{Y}^T)) = \frac{1}{n}\,\mathrm{sl}(\underline{\underline{H}}\,\mathrm{var}(\underline{X}))$$

i) Če so $X_1, X_2, \dots, X_n$ nekorelirane, je $\mathrm{var}(\underline{X}) = \sigma^2 \underline{\underline{I}}_n$ in $E(\hat{\sigma}^2) = \frac{1}{n}\,\mathrm{sl}(\underline{\underline{H}})\sigma^2 = \frac{1}{n}\,\mathrm{sl}(\underline{\underline{I}}_n - \frac{1}{n}\underline{1}\,\underline{1}^T)\sigma^2 = \frac{1}{n}(n-\frac{1}{n})\sigma^2 = \frac{n-1}{n}\sigma^2$.

Popravljena vzorčna varianca je $\hat{\sigma}_+^2 = \frac{n}{n-1}\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2$ pa je nepristranska do σ^2.

Opomba: $\hat{\sigma}_+$ pa je tipično pristranska cenilka za σ!

ii) Enostavno slučajno vzorčenje:

Verjetno bomo delali na vajah ...

Spomnimo se še, da je standardna napaka nepristranske cenilke $\overline{X}$ za μ enaka $SE = \frac{\sigma}{\sqrt{n}}$.

$\hat{SE} = \frac{\hat{\sigma}}{\sqrt{n}}$ je cenilka za SE.

$\hat{SE}^2 = \frac{\hat{\sigma}^2}{n} = \frac{1}{n^2}\sum_{i=1}^{n}(X_i-\overline{X})^2$ je cenilka za SE^2.

$\hat{SE}_+^2 = \frac{\hat{\sigma}_+^2}{n} = \frac{1}{n(n-1)}\sum_{i=1}^{n}(X_i-\overline{X})^2$ je nepristranska cenilka za SE^2.

Metoda momentov je poseben primer metode empirične porazdelitve:

Teoretični moment $m_k = E(X_1^k)$ ocenimo z empiričnim momentom:

$$\hat{m}_k = \frac{X_1^k + X_2^k + \dots + X_n^k}{n} = m_k(\mathrm{Emp}(X_1, X_2, \dots, X_n))$$

Preučevano količino y izrazimo z momenti $y = g(m_1, m_2, \dots, m_k)$ in ocenimo z $\hat{y} = g(\hat{m}_1, \hat{m}_2, \dots, m_k)$.

Primer: $\sigma^2 = m_2 - m_1^2$

$$\hat{\sigma}^2 = \hat{m}_2 - \hat{m}_1^2 = \frac{1}{n}\sum X_i^2 - \left(\frac{1}{n}\sum X_i\right)^2 = \frac{1}{n}\sum (X_i - \overline{X})^2$$

METODA NAJVEČJEGA VERJETJA

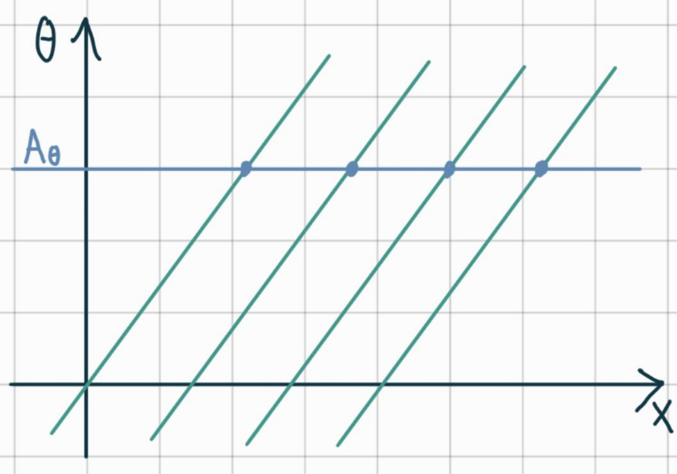
Definicija: Verjetje (angl. likelihood) je verjetnostna funkcija ali gostota opažanja, ki jo gledamo kot funkcijo parametra modela.

Naj opažanje zavzame vrednosti v množici A, neodvisni od parametra θ in privzemimo še enega od naslednjih dveh pogojev:

(D) A je števna in verjetje $L(\theta | x) = P_\theta(X = x) > 0$ za vse $x \in A$.

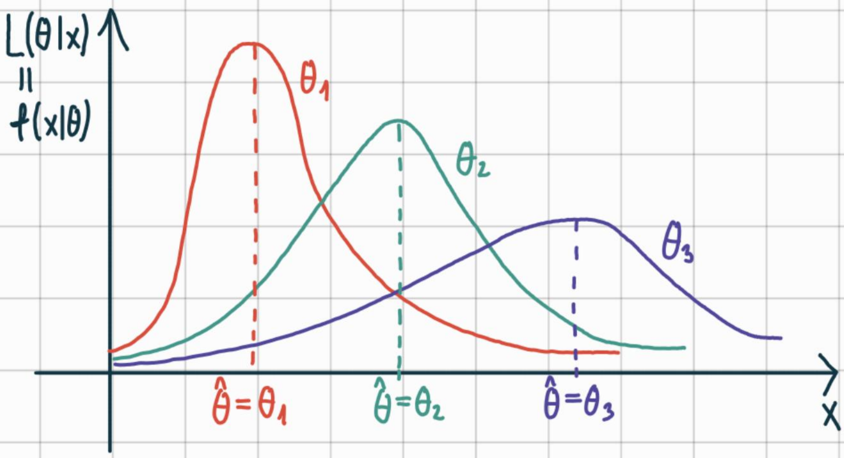
(Z) A je odprta podmnožica prostora $\mathbb{R}^n$, opažanje $\underline{X}$ je porazdeljeno zvezno in $L(\theta | \underline{x}) = f_{\underline{X}}(\underline{x} | \theta) > 0$ za vse $\underline{x} \in A$.

Opomba: Kaj ne sme veljati?



Definicija: Cenilka $\hat{\theta} = h(X)$ za θ **po metodi največjega verjetja (CMNV)** je tista cenilka, pri kateri $L(\theta|x)$ pri vsakem $x \in A$ doseže maksimum pri $\theta = h(x)$.

Primer: $\Theta = \{\theta_1, \theta_2, \theta_3\}$



Če je $\hat{\theta}$ cenilka za θ, je $g(\hat{\theta})$ cenilka za $g(\theta)$.

Namesto L radi minimiziramo $\ell = \ln L$.

Opažanje je često vektor iz neodvisnih in enako porazdeljenih slučajnih spremenljivk $X_1, X_2, \dots, X_n$.

Verjetje za posamezno opažanje:

$$L_1(\theta|x) = P_\theta(X_1 = x) \text{ ali } f_{X_1}(x|\theta)$$

$$L(\theta|x_1, x_2, \dots, x_n) = L_1(\theta|x_1) L_1(\theta|x_2) \cdots L_1(\theta|x_n)$$

$$\ell_1(\theta|x) := \ln L(\theta|x)$$

$$\ell(\theta|x_1, x_2, \dots, x_n) = \ell_1(\theta|x_1) + \ell_1(\theta|x_2) + \dots + \ell_1(\theta|x_n)$$

Primer: Opazimo $X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2)$ neodvisne, $\sigma > 0$.

$$\Theta = \mathbb{R} \times (0, \infty)$$

$\underline{\theta} = (\mu, \sigma) \in \Theta$

$$L_1(\underline{\theta} | x) = \frac{1}{\sigma \sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$\parallel$

$L_1(\mu, \sigma | x)$

$$\ell_1(\mu, \sigma | x) = -\ln \sigma - \ln \sqrt{2\pi} - \frac{(x-\mu)^2}{2\sigma^2}$$

$$\ell(\mu, \sigma | x_1, x_2, \dots, x_n) = -n \cdot \ln \sigma - n \cdot \ln \sqrt{2\pi} - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2$$

Fiksirajmo $x_1, x_2, \dots, x_n$. Pri katerih μ in σ je to maksimalno?

$$\frac{\partial \ell}{\partial \mu} = -\frac{1}{2\sigma^2} \cdot 2 \sum_{i=1}^{n} (x_i - \mu) = \frac{1}{\sigma^2} \cdot \left(\sum_{i=1}^{n} x_i - n\mu\right) = 0$$

$$\Rightarrow \mu = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\Rightarrow \hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$\frac{\partial \ell}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (x_i - \mu)^2 = 0 \qquad / \cdot \sigma^3$$

$$\Rightarrow n\sigma^2 = \sum_{i=1}^{n} (x_i - \mu)^2$$

$$\Rightarrow \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{x})^2 , \quad \overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \hat{\mu})^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

V tem primeru (pri splošni porazdelitvi pa morda ne) dobimo isti cenilki kot pri metodi empirične porazdelitve.

$\Theta \subseteq \mathbb{R}^p$ odprta

$\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_p) \in \Theta$

Zbirna funkcija (angl. score function):

$$\underline{s}(\underline{\theta}|x) := \nabla_{\underline{\theta}} \ell(\theta|x) = \begin{bmatrix} \frac{\partial \ell}{\partial \theta_1} \\ \vdots \\ \frac{\partial \ell}{\partial \theta_p} \end{bmatrix} = \frac{\nabla_{\underline{\theta}} L(\underline{\theta}|x)}{L(\underline{\theta}|x)}$$

Za $g: A \to \mathbb{R}$ definiramo:

$$\int_A g(x)\, m(dx) := \begin{cases} \sum_{x \in A} g(x) & ; \text{ primer (D)} \\ \int_A g(\underline{x})\, d\underline{x} & ; \text{ primer (Z)} \end{cases}$$

$$\int_A L(\underline{\theta}|x)\, m(dx) = 1 \quad \Big/ \frac{\partial}{\partial \theta_j}$$

(01): $L(\underline{\theta}|x)$ je parcialno zvezno odvedljiva po $\underline{\theta}$ in za vsak $\theta^* \in \Theta$ obstaja $U \overset{odp}{\subseteq} \Theta$, $\theta^* \in U$, da za vsak j velja:

$$\int_A \sup_{\theta \in U} \left| \frac{\partial L}{\partial \theta_j}(\theta|x) \right| m(dx) < \infty$$

$$\overset{(01)}{\Rightarrow} \int_A \frac{\partial L}{\partial \theta_j}(\underline{\theta}|x)\, m(dx) = 0$$

$$\int_A \frac{\partial L}{\partial \theta_j}(\underline{\theta}|x)\, m(dx) = \int_A \frac{\partial \ell}{\partial \theta_j}(\underline{\theta}|x)\, L(\underline{\theta}|x)\, m(dx) = E_\theta\left[\frac{\partial \ell}{\partial \theta_j}(\underline{\theta}|X)\right]$$

Opazka 6.1: Pri pogoju (01) je $E_{\underline{\theta}}[\underline{s}(\underline{\theta}|X)] = 0$.

$$\frac{\partial^2 \ell}{\partial \theta_j \partial \theta_k}(\underline{\theta}|x) = \frac{\partial}{\partial \theta_k} \frac{\frac{\partial L}{\partial \theta_j}(\underline{\theta}|x)}{L(\underline{\theta}|x)}$$

$$= \frac{\frac{\partial^2 L}{\partial \theta_j \partial \theta_k}(\underline{\theta}|x) \cdot L(\underline{\theta}|x) - \frac{\partial L}{\partial \theta_j}(\underline{\theta}|x) \cdot \frac{\partial L}{\partial \theta_k}(\underline{\theta}|x)}{(L(\underline{\theta}|x))^2}$$

$$= \frac{\frac{\partial^2 L}{\partial \theta_j \partial \theta_k}(\underline{\theta}|x)}{L(\underline{\theta}|x)} - \frac{\partial \ell}{\partial \theta_j}(\underline{\theta}|x) \cdot \frac{\partial \ell}{\partial \theta_k}(\underline{\theta}|x)$$

Imamo še močnejši pogoj:

(O2): $L(\underline{\theta}|x)$ je dvakrat parcialno zvezno odvedljiva po $\underline{\theta}$ in za vsak $\theta^* \in \Theta$ obstaja odprta okolica $U^{odp} \subseteq \Theta$ od θ^*, da za vse j in k velja:

$$\int_A \sup_{\theta \in U} \left| \frac{\partial^2 L}{\partial \theta_j \partial \theta_k}(\underline{\theta}|x) \right| m(dx) < \infty$$

Če velja še (O2), dobimo:

$$\frac{\partial^2}{\partial \theta_j \partial \theta_k} \int_A L(\underline{\theta}|x)\, m(dx)$$

$$= \int_A \frac{\partial^2 L}{\partial \theta_j \partial \theta_k}(\underline{\theta}|x)\, m(dx)$$

$$= E_{\underline{\theta}}\left[\frac{\frac{\partial^2 L}{\partial \theta_j \partial \theta_k}(\underline{\theta}|X)}{L(\underline{\theta}|X)}\right] = 0$$

$$E_{\underline{\theta}}\left[\frac{\partial^2 \ell}{\partial \theta_j \partial \theta_k}(\underline{\theta}|X)\right] = -E_{\underline{\theta}}\left[\frac{\partial \ell}{\partial \theta_j}(\underline{\theta}|X) \cdot \frac{\partial \ell}{\partial \theta_k}(\underline{\theta}|X)\right]$$

opazka 6.1

$$= -\operatorname{cov}_{\theta}\left(\frac{\partial \ell}{\partial \theta_j}(\underline{\theta}|X), \frac{\partial \ell}{\partial \theta_k}(\underline{\theta}|X)\right)$$

Hessejeva matrika: $\nabla^2 g := \left[\frac{\partial^2 g}{\partial \theta_j \partial \theta_k}\right]_{j,k}$

Opazka 6.2: Pod pogojema (O1) in (O2) velja:

$$\operatorname{Var}_{\underline{\theta}}(s(\underline{\theta}|X)) = -E_{\underline{\theta}}\left[\nabla^2_{\underline{\theta}} \ell(\underline{\theta}|X)\right]$$

Definicija: Fisherjeva informacija je matrika:

$$FI(\theta_1, \theta_2, \dots, \theta_n) := \operatorname{Var}_{\theta}[s(\underline{\theta}|X)] = -E_{\underline{\theta}}\left[\nabla^2_{\underline{\theta}} \ell(\underline{\theta}|X)\right]$$

Naj bo $X = \underline{X} = (X_1, X_2, \dots, X_n)$, kjer so $X_1, X_2, \dots, X_n$ neodvisne in enako porazdeljene.

Definirajmo $FI_1(\underline{\theta}) := -E[\nabla^2_{\underline{\theta}} \ell_1(\underline{\theta}|X_1)]$.

Opazka 6.3: $FI(\underline{\theta}) = n \cdot FI_1(\underline{\theta})$

$$\left(= -E[\nabla^2_{\underline{\theta}} \ell(\underline{\theta}|X_1, X_2, \dots, X_n)]\right)$$

Izrek 6.4: Opazimo neodvisne in enako porazdeljene $X_1, X_2, \dots, X_n$.

Naj bo $\Theta \overset{odp}{\subseteq} \mathbb{R}^p$ in $\underline{\theta}^* \in \Theta$

Naj bo $FI_1(\underline{\theta}^*)$ obrnljiva.

Naj veljata (O1) in (O2).

Naj bo $\hat{\underline{\theta}}$ cenilka za θ po MNV, kadar je slednja nedvoumno definiranja, sicer pa poljubna, da je le $E_{\underline{\theta}^*}(\|\hat{\underline{\theta}}\|^2) < \infty$.

Tedaj velja:

1) $\hat{\underline{\theta}} \xrightarrow[n\to\infty]{d, P_{\theta^*}} \underline{\theta}^*$

2) $\sqrt{n} \cdot Bias_{\underline{\theta}^*}(\hat{\underline{\theta}}|\underline{\theta}^*) \xrightarrow[n\to\infty]{} 0$

3) $n \cdot Var_{\underline{\theta}^*}(\hat{\underline{\theta}}) \longrightarrow (FI_1(\underline{\theta}^*))^{-1}$

Oziroma: $Var_{\underline{\theta}^*}(\hat{\underline{\theta}}) \sim (FI(\underline{\theta}^*))^{-1}$

4) $n \cdot E_{\underline{\theta}^*}[(\hat{\underline{\theta}} - \underline{\theta}^*)(\hat{\underline{\theta}} - \underline{\theta}^*)^T] \longrightarrow (FI_1(\underline{\theta}^*))^{-1}$

Poleg tega za primerno funkcijo g in cenilko $g(\hat{\theta}) = \hat{\gamma}$ za $\gamma = g(\theta)$ velja:

1') $\hat{y} \xrightarrow[n\to\infty]{P_{\underline{\theta}^*},\, d} y$

2') $\sqrt{n}\cdot \mathrm{Bias}_{\underline{\theta}^*}(\hat{y}\,|\,y) \xrightarrow[n\to\infty]{} 0$

3') $n\cdot \mathrm{var}_{\underline{\theta}^*}(\hat{y}) \xrightarrow[n\to\infty]{} (\nabla g(\underline{\theta}^*))^T \, (FI_1(\underline{\theta}^*))^{-1} \, (\nabla g(\underline{\theta}^*))$

4') $n\cdot MSE_{\underline{\theta}^*}(\hat{y}\,|\,y) \xrightarrow[n\to\infty]{}$

Dokaz opustimo.