NAVADNE DIFERENCIALNE ENAČBE

NDE n-tega reda: $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$

NDE 1. reda: $y' = f(x,y)$

sistem NDE n-tega reda: $Y' = F(x, Y, Y', \dots, Y^{(n-1)})$

sistem NDE 1. reda: $Y' = F(x, Y)$, $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, $F = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$

Rešitev NDE je n-parametrična družina funkcij.

ZAČETNI PROBLEM

Izberemo tisto rešitev y iz družine rešitev, ki zadošča začetnemu pogoju:

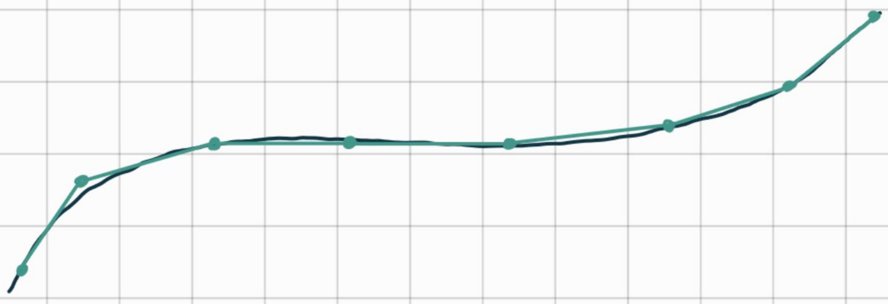
$$y(0) = y_{0,0}$$
$$y'(0) = y_{0,1}$$
$$\vdots$$
$$y^{(n-1)}(0) = y_{0,n-1}$$

EKSISTENČNI IZREK

(Picard-Lindelöf)

Obstaja enolična rešitev začetnega problema $y' = f(x,y)$, $y(0) = y_0$, če je f zvezna v x in Lipschitzeva v y.

NUMERIČNO REŠEVANJE



Iščemo $y_i \approx y(x_i)$.

EULERJEVA METODA

$$y_0 = y(0)$$

$$y' = F(x,y)$$

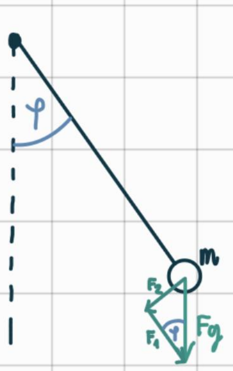
$$y_{i+1} = y_i + \underbrace{h \cdot F(x_i, y_i)}_{\text{premik v smeri tangente}}, \quad i = 0, 1, \ldots$$

METODE RUNGE-KUTTA

Razred metod, kjer posodobimo tangentno smer večkrat tekom koraka.

PRIMERI MODELIRANJA

MATEMATIČNO NIHALO



2. Newtonov zakon: $F_2 = m \cdot (-l \cdot \ddot{\varphi})$

$\sin\varphi \cdot \cancel{m} \cdot g = -\cancel{m} \cdot l \cdot \ddot{\varphi}$

$\ddot{\varphi} = -\frac{l}{g} \cdot \sin\varphi \quad , \quad \varphi(0) = \varphi_0, \; \dot{\varphi}(0) = 0$

Rešitev ni elementarna funkcija.

1. pristop: Model dodatno poenostavimo

$\varphi \ll 1 \Rightarrow \sin\varphi \approx \varphi$

$\Rightarrow \ddot{\varphi} = -\frac{l}{g} \cdot \varphi$

$\Rightarrow$ Rešimo analitično

Nastavek: $\lambda^2 = -\frac{g}{l}$, ničli λ_1, λ_2

$\Rightarrow \varphi(t) = Ae^{i\lambda_1 t} + Be^{i\lambda_2 t}$

Začetni pogoji $\Rightarrow \varphi(t) = \varphi_0 \cdot \cos\left(\sqrt{\frac{g}{l}} \cdot t\right)$

2. pristop: Numerično reševanje

$$Y = \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

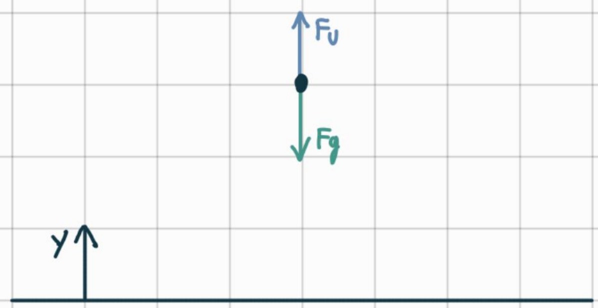
$$\begin{aligned} \varphi &= y_1 \\ \dot{\varphi} &= \dot{y}_1 = y_2 \\ \ddot{\varphi} &= \ddot{y}_1 = \dot{y}_2 = -\frac{g}{l} \sin(y_1) \end{aligned}$$

$$F(t, Y) = \dot{Y} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -\frac{g}{l} \sin(y_1) \end{bmatrix}$$

$$Y_0 = \begin{bmatrix} \varphi_0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Rešimo numerično

PROSTI PAD PADALCA



$y(t)$ ... višina padalca
$\dot{y}(t)$ ... hitrost padalca
$\ddot{y}(t)$ ... pospešek padalca

2. Newtonov zakon: $\quad ma = F_g + F_u$

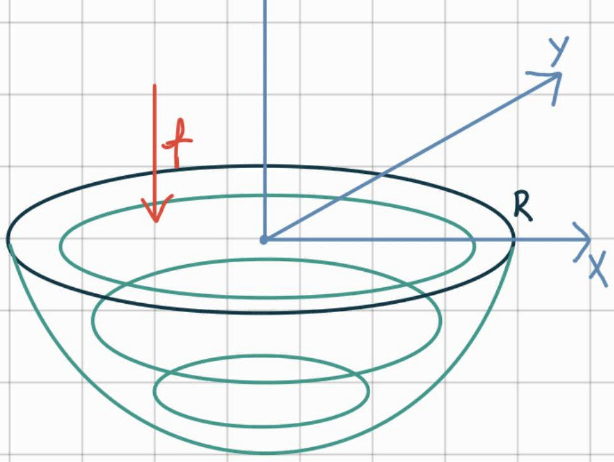
$$m\ddot{y} = -mg + \frac{S \cdot \rho_z \cdot C_u}{2} \cdot |\dot{y}| \cdot \dot{y}$$

$$\ddot{y} = -g + \frac{S \cdot \rho_z \cdot C_u}{2m} \cdot |\dot{y}| \cdot \dot{y}$$

S ... prečni presek telesa
ρ_z ... gostota zraka
C_u ... koeficient zračnega upora

$g \approx 9{,}81\ m/s^2$ (blizu površja)
$\rho_z \approx 1{,}2\ kg/m^3$ (blizu površja)
$C_u \approx 1$

TANKA OPNA NAD KROŽNO ZANKO



D ... domena (notranjost kroga z radijem R

∂D ... rob domene (krožnica z radijem R)

f ... zunanja sila

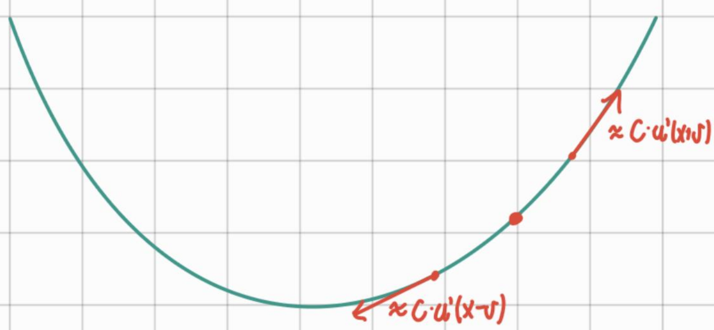
Iščemo $u: \partial D \cup D \to \mathbb{R}$, ki opisuje vertikalni upogib opna.

Izkaže se, da za majhne upogibe velja:

$$-\Delta u(x,y) = f(x,y) \quad , \quad (x,y) \in D$$
$$u(x,y) = 0 \quad , \quad (x,y) \in \partial D$$

... robni problem

Δu ... navpična sila napetosti opne



Skupna sila napetosti:

$$C \cdot u'(x+\sigma) + C \cdot u'(x-\sigma) \approx C \cdot u''(x)dx$$

Predpostavimo, da je f radialno simetrična (neodvisna od kota) in opna homogena.

Vpeljemo polarne koordinate:

$x = r\cos\varphi$
$y = r\sin\varphi$

$r \in [0, R]$
$\varphi \in [0, 2\pi)$

$$\Delta u(r,\varphi) \overset{\text{verižno pravilo}}{=} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \varphi^2}$$

Torej rešujemo:

$$u''(r) + \frac{1}{r}\cdot u'(r) = f(r) \quad , \quad r \in [0,R]$$

Robni pogoji:

$u(R) = 0$
$u'(0) = 0$ (simetrija)

Rešujemo z uporabo metode končnih diferenc.

Ideja: Odvode aproksimiramo s končnimi diferencami v vnaprej predpisanih točkah domene in tako prevedemo diferencialno enačbo na sistem algebraičnih enačb.

$h = r_{i+1} - r_i$

Neznanke so $u_0, u_1, \dots, u_{n-1}$.

$$r_i = \frac{i \cdot R}{n} \quad , \quad u_i \approx u(r_i)$$

Enačbo $u''(r) + \frac{1}{r} u'(r) = f(r)$ poglejmo samo v r_i:

$$u''(r_i) + \frac{1}{r_i} u'(r_i) = f(r_i), \quad i = 1,2,\dots,n-1$$

Odvode zamenjamo s končnimi diferencami.

Iz Taylorja dobimo:

$$(1): \underbrace{u(r_i+h)}_{\approx u_{i+1}} = \underbrace{u(r_i)}_{\approx u_i} + h \cdot u'(r_i) + \frac{h^2}{2} \cdot u''(r_i) + O(h^3)$$

$$(2): \underbrace{u(r_i-h)}_{\approx u_{i-1}} = \underbrace{u(r_i)}_{\approx u_i} - h \cdot u'(r_i) + \frac{h^2}{2} \cdot u''(r_i) + O(h^3)$$

$$(1)-(2): \underbrace{\frac{u_{i+1}-u_{i-1}}{2h}}_{\text{simetrična diferenca}} = u'(r_i) + O(h^2)$$

$$(1)+(2): \frac{u_{i+1}-2u_i+u_{i-1}}{h^2} = u''(r_i) + O(h^2)$$

$$u''(r_i) + \frac{1}{r_i} u'(r_i) = f(r_i)$$

$$\Rightarrow \left(1-\frac{1}{i}\right) u_{i-1} - 2u_i + \left(1+\frac{1}{i}\right) u_{i+1} = h^2 f_i$$

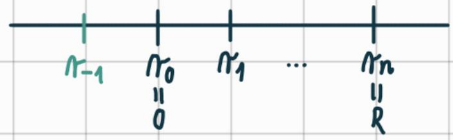
$$r_i = ih \Rightarrow \frac{h}{r_i} = \frac{h}{ih} = \frac{1}{i}$$

$$\Rightarrow i = n-1: \left(1-\frac{1}{i}\right) u_{i-1} - 2u_i + \overbrace{\left(1+\frac{1}{i}\right) \underbrace{u_n}_{=0}}^{=0} = h^2 f_i$$

Robni pogoj: $u'(0) = 0$

i) $u'(0) = \frac{u_1-u_2}{h} + O(h)$:(

ii) Dodamo navidezno točko r_{-1}:



$$0 = u'(0) = \frac{u_1 - u_{-1}}{2h} + O(h^2)$$

$$\Rightarrow u_1 = u_{-1}$$

$u_1 = u_{-1}$ vstavimo v enačbo pri $i = 0$:

$$\left(1 - \cancel{\tfrac{1}{0}}\right) u_1 - 2u_0 + \left(1 + \cancel{\tfrac{1}{0}}\right) u_1 = h^2 \cdot f_0$$

$$-2u_0 + 2u_1 = h^2 \cdot f_0$$

Dobimo n enačb in n neznank.

$$\underbrace{\begin{bmatrix} -2 & 1+\frac{1}{1} & & & \\ 1-\frac{1}{2} & -2 & 1+\frac{1}{2} & & \\ & 1-\frac{1}{3} & -2 & 1+\frac{1}{3} & \\ & & \ddots & \ddots & \ddots \\ & & & 1-\frac{1}{n-1} & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_{n-1} \end{bmatrix}}_{\underline{u}} = \underbrace{\begin{bmatrix} h^2 f_0 \\ h^2 f_1 \\ \vdots \\ \vdots \\ h^2 f_{n-1} \end{bmatrix}}_{\underline{f}}$$

Dobimo 3-diagonalen sistem, ki je šibko diagonalno dominanten po vrsticah:

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

$$|a_{n-1,n-1}| > \sum_{\substack{j=1 \\ j \neq n-1}}^{n} |a_{n-1,j}|$$

Hkrati velja še, da je ta matrika nerazcepna.

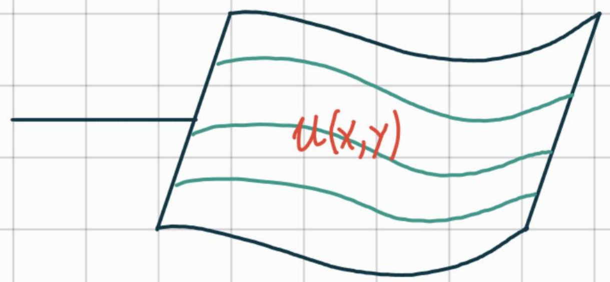
Sledi, da je A obrnljiva, prav tako vse njene glavne podmatrike.

Torej obstaja LU razcep brez pivotiranja.

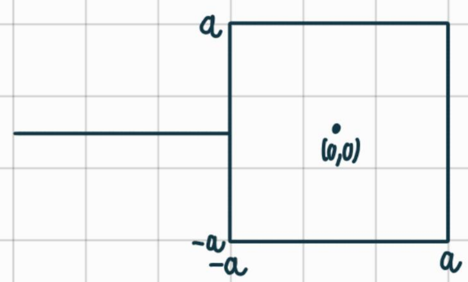
Lahko uporabimo Thomasov algoritem za 3-diagonalne matrike.

MILNI MEHURČEK NAD ŽIČNO ZANKO

Žično zanko potopimo v milnico. Zanima nas oblika milnega mehurčka.



Poenostavimo obliko zanke, da iz vrha izgleda kot kvadrat.



Obliko mehurčka opisuje Young-Laplaceova enačba:

$$\Delta p = -2\gamma \cdot H$$

površinska napetost

tlačna razlika mehurčka

površinska napetost

$$H := -\frac{1}{2} \underbrace{\nabla}_{\text{divergenca}} \cdot \underbrace{\left(\frac{\nabla U}{\|\nabla U\|}\right)}_{\text{norma ploskve}} = *$$

$$U(x,y,z) := z - u(x,y)$$

$$* = -\frac{1}{2} \nabla \cdot \frac{(U_x, U_y, U_z)}{\|(U_x, U_y, U_z)\|} = -\frac{1}{2} \nabla \cdot \frac{(-u_x, -u_y, 1)}{\underbrace{\sqrt{1+u_x^2+u_y^2}}_{N}} = *$$

$$N_x = \frac{1}{N}\left(u_x \cdot u_{xx} + u_y \cdot u_{xy}\right)$$
$$N_y = \frac{1}{N}\left(u_y \cdot u_{yy} + u_x \cdot u_{xy}\right)$$

$$* = \frac{1}{2} \cdot \frac{(u_{xx} \cdot N - u_x \cdot N_x) + (u_{yy} \cdot N - u_y \cdot N_y)}{N^2} = \frac{1}{2} \cdot \frac{u_{xx}(1+u_y^2) - 2u_x u_y u_{xy} + u_{yy}(1+u_x^2)}{(1+u_x^2+u_y^2)^{3/2}}$$

Dobimo nelinearno PDE!

Zanemarimo tlačno razliko ($\Delta p = 0$).

$$\Rightarrow u_{xx}(1+u_y^2) - 2u_x u_y u_{xy} + u_{yy}(1+u_x^2) = 0$$

Rešitve problema so t.i. minimalne ploskve. To so ploskve, za katere velja, da imajo minimalno površino. Hkrati vemo, da za njih velja $H \equiv 0$.

Linearizacija PDE:

$\|\nabla u\| \ll 1$

$$\Rightarrow \sqrt{1+\|\nabla u\|_2^2} \approx 1 + \frac{1}{2} \cdot \frac{1}{\sqrt{1+\|\nabla u\|_2^2}}\Big|_{\|\nabla u\|_2=0} \cdot \|\nabla u\|_2 = 1 + \frac{1}{2}\|\nabla u\|_2$$

$$\Rightarrow H \approx \frac{1}{2} \nabla \cdot \frac{\nabla u}{1+\frac{1}{2}\|\nabla u\|_2} \approx \frac{1}{2} \nabla \cdot \nabla u = \frac{1}{2} \Delta u$$

Poenostavljen model mehurčka:

$$\Delta u(x,y) \overset{*}{=} 0, \quad (x,y) \in (-a,a)^2$$

$$u(x,y) = f(x,y), \quad (x,y) \in \partial((-a,a)^2)$$

$$-a = x_0 < x_1 < \ldots < x_{n+1} = a$$
$$-a = y_0 < y_1 < \ldots < y_{n+1} = a$$

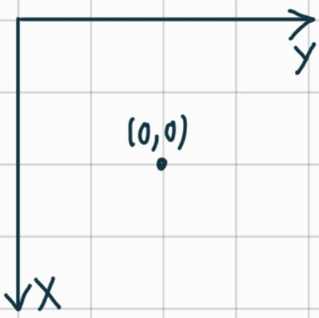
Korak: $h = x_{i+1} - x_i = y_{i+1} - y_i$

Zanimajo nas vrednosti $u_{ij} \approx u(x_i, y_j)$ za $i,j = 1,\ldots,n$.

Robni pogoji:

$$u_{i,0} = f(x_i, -a) \qquad u_{0,j} = f(-a, y_j)$$
$$u_{i,n+1} = f(x_i, a) \qquad u_{n+1,j} = f(a, y_j)$$

V Matlabu obrnemo koordinatni sistem:



Končne diference v notranjosti domene:

$$\frac{\partial^2 u(x_i,y_j)}{\partial x^2} = \frac{u_{i-1,j} - 2u_{ij} + u_{i+1,j}}{h^2} + \mathcal{O}(h^2)$$

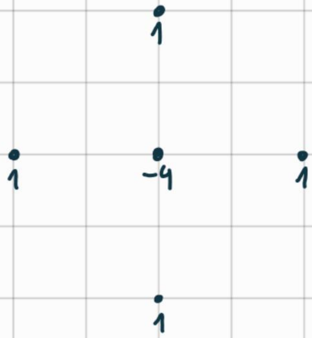
$$\frac{\partial^2 u(x_i,y_j)}{\partial y^2} = \frac{u_{i,j-1} - 2u_{ij} + u_{i,j+1}}{h^2} + \mathcal{O}(h^2)$$

$$\Delta u(x_i, y_j) = \frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j) \overset{*}{=} 0$$

Dobimo naslednji sistem enačb:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{ij} = 0 \quad , \quad i,j = 1,\dots,n$$

... 5-točkovna shema



Rešiti moramo sistem $Au = b$ z razpršeno matriko A.

Če to delamo s polnimi matrikami, potrebujemo $\mathcal{O}(n^4)$ prostora za shranjevanje in $\mathcal{O}(n^6)$ časa za LU razcep.

Namesto tega uporabimo iterativno reševanje.

$A u = b \quad , \quad A = M + N$

$(M+N) u = b$

$M u^{(i)} = b - N u^{(i-1)} \quad , \quad i = 1, 2, \ldots$

$u^{(0)}$ je začetni približek za u.

Vsak korak rešimo linearen sistem z matriko M.

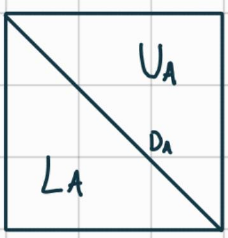
Želimo, da $u^{(i)}$ konvergira k u.

Želimo izbrati lepo matriko M, da bo računanje enostavno.

Največkrat vzamemo diagonalno ali trikotno matriko.

Jacobijeva iteracija (J): $\quad M = D_A \; , \; N = L_A + U_A$

Gauss-Seidlova iteracija (GS): $\quad M = D_A + L_A \; , \; N = U_A$



Trditev: Iteracija $u^{(i)}$ konvergira k rešitvi u za vsak začetni $u^{(0)}$, če velja $\rho(M^{-1}N) < 1$ (spektralni radij).

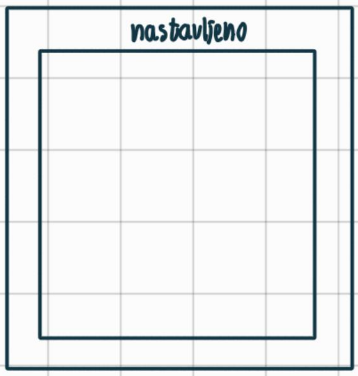
Opomba: $\rho(A) = \max_i |\lambda_i(A)| \; , \; \lambda_i$ lastne vrednosti A

Da se pokazati, da J in GS konvergirata za naš problem, saj je A šibko diagonalno dominantna s strogo neenakostjo v vsaj eni vrstici in nerazcepna, zato je $\rho(M^{-1}N) < 1$.

Lahko uporabimo tudi iterativno reševanje direktno na mreži:

- Nastavimo in fiksiramo robne vrednosti v matriki $U \in \mathbb{R}^{(n+2)\times(n+2)}$:

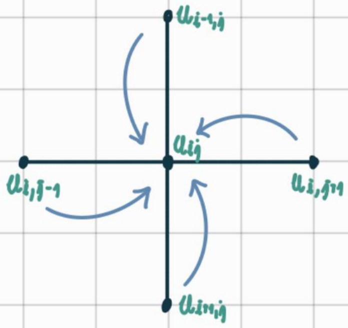
rešitev

$U =$ 

- En korak J pomeni izračunavanje povprečja notranjih elementov na podlagi 4 sosedov:

$$u_{i,j-1} + u_{i,j+1} + u_{i-1,j} + u_{i+1,j} - 4u_{ij} = 0$$

$$\Rightarrow u_{ij} = \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}}{4}$$



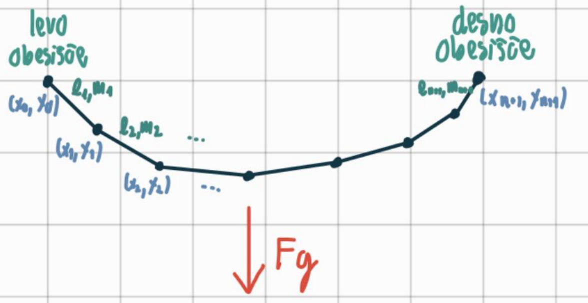
- En korak GS deluje podobno, le da pri računanju povprečja že upoštevamo posodobljene vrednosti trenutnega koraka

V tem primeru je prostorska zahtevnost $\mathcal{O}(n^2)$, računska zahtevnost pa $\mathcal{O}(k \cdot n^2)$, kjer je k število iteracij.

DISKRETNA VERIŽNICA

n+1 členkov (ravnih togih palic) dolžin l_i in mas m_i vpnemo med seboj zaporedno z gibljivimi stiki, skrajni točki pa fiksiramo (obesišči).

Dobljena verižnica je opisana s koordinatami (x_i, y_i), $i = 0, 1, \dots, m+1$.



Minimizirati želimo potencialno energijo mgh:

$$F(\underline{x}, \underline{y}) := \sum_{i=1}^{n+1} \underbrace{m_i \cdot \frac{y_i + y_{i-1}}{2}}_{\text{potencialna energija } i\text{-tega členka}}$$

Kjer sta:

$$\underline{x} := (x_0, \dots, x_{n+1})$$
$$\underline{y} := (y_0, \dots, y_{n+1})$$

Pri pogojih:

$$l_i^2 = (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2, \quad i = 1, 2, \dots, n+1$$

Označimo:

$$l_0 := \|(x_{n+1}, y_{n+1}) - (x_0, y_0)\|_2$$

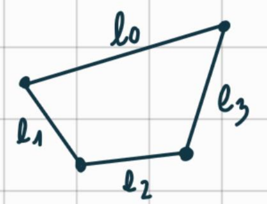
Predpogoj 1: Poligonska neenakost:

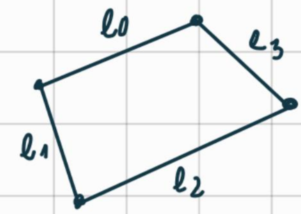
$$l_i < \sum_{\substack{j=1 \\ j \neq i}}^{n+1} l_j, \quad i = 0, 1, \dots, n+1$$

Predpogoj 2: $l_i < l_0$, $i = 0, 1, \dots, n+1$

Sledi: Dobljena verižnica graf odsekoma linearne konveksne funkcije

Primer:





$l_1, l_2, l_3 < l_0$ ✓

$l_2 > l_0$ X

Vezani ekstrem določimo s pomočjo Lagrangeevih multiplikatorjev:

$$\nabla F = -\lambda \nabla g$$

$$G(\underline{x}, \underline{y}, \underline{\lambda}) := \sum_{i=1}^{n+1} \left[m_i \cdot \frac{y_i - y_{i-1}}{2} + \lambda_i \cdot \left((x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 - l_i^2 \right) \right]$$

Dobimo $3n+1$ enačb za stacionarno točko (ekstrem) funkcije G:

$$\frac{\partial G}{\partial x_i} = 0 \quad , \quad i = 1, \dots, n$$
$$\frac{\partial G}{\partial y_i} = 0 \quad , \quad i = 1, \dots, n$$
$$\frac{\partial G}{\partial \lambda_i} = 0 \quad , \quad i = 1, \dots, n+1$$

$$\Delta x_i = x_i - x_{i-1}$$
$$\Delta y_i = y_i - y_{i-1}$$

$$\Rightarrow x_i = x_0 + \sum_{j=1}^{i} \Delta x_j \qquad (***)$$
$$y_i = y_0 + \sum_{j=1}^{i} \Delta y_j$$

Torej:

$$\frac{\partial G}{\partial x_i} \propto \lambda_i \Delta x_i - \lambda_{i+1} \Delta x_{i+1} = 0 \qquad (1)$$

$$\frac{\partial G}{\partial y_i} \propto \lambda_i \Delta y_i - \lambda_{i+1} \Delta y_{i+1} = -\underbrace{\frac{m_i + m_{i+1}}{4}}_{=: \frac{\mu_i}{2}} \qquad (2)$$

$$\frac{\partial G}{\partial \lambda_i} \propto \Delta x_i^2 + \Delta y_i^2 = l_i^2 \qquad (3)$$

(1) $\Rightarrow$ $\lambda_i \Delta x_i$ so vsi enaki

Označimo: $\lambda_i \Delta x_i =: -\frac{1}{2u}$ za neznanko u

Oziroma: $\lambda_i = -\frac{1}{2u \underset{\text{(predpogoj 2)}}{\underset{0}{\underset{\vee}{\Delta x_i}}}}$

Vstavimo v (2):

$$\frac{1}{2u} \cdot \frac{\Delta y_i}{\Delta x_i} - \frac{1}{2u} \cdot \frac{\Delta y_{i+1}}{\Delta x_{i+1}} = \frac{\mu_i}{2}$$

$$\frac{\Delta y_{i+1}}{\Delta x_{i+1}} = \frac{\Delta y_i}{\Delta x_i} - u\mu_i \quad , \quad i = 1, 2, \dots, n$$

$\downarrow$

$$\frac{\Delta y_{i+1}}{\Delta x_{i+1}} = \frac{\Delta y_{i-1}}{\Delta x_{i-1}} - u\mu_{i-1} - u\mu_i$$

$\downarrow$
$\downarrow$

$$\frac{\Delta y_i}{\Delta x_i} = \underbrace{\frac{\Delta y_1}{\Delta x_1}}_{=: v} - u\sum_{j=1}^{i-1} \mu_j \quad , \quad i = 1, 2, \dots, n+1$$

Skupaj:

$$v = \frac{\Delta y_1}{\Delta x_1} \quad \text{in} \quad \frac{\Delta y_i}{\Delta x_i} = v - u\sum_{j=1}^{i-1} \mu_j \quad , \quad i = 1, 2, \dots, n+1 \qquad (*)$$

(3) $\Rightarrow$ $1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2 = \left(\frac{l_i}{\Delta x_i}\right)^2 \quad , \quad i = 1, 2, \dots, n+1$

$$\Rightarrow 1 + \left(v - u\sum_{j=1}^{i-1} \mu_i\right)^2 = \left(\frac{l_i}{\Delta x_i}\right)^2$$

$$\Rightarrow \Delta x_i(u, v) = \frac{l_i}{\sqrt{1 + (v - u\sum \mu_i)^2}} \quad , \quad i = 1, 2, \dots, n+1 \qquad (**)$$

Ko enkrat poznamo Δx_i, bomo dobili Δy_i iz $(*)$.

Zadeti moramo zadnjo točko (x_{n+1}, y_{n+1}):

$$U(u,v) := \sum_{j=1}^{n+1} \Delta x_j(u,v) - (x_{n+1} - x_0)$$

$$V(u,v) := \sum_{j=1}^{n+1} \Delta y_j(u,v) - (y_{n+1} - y_0)$$

Poiskati moramo u, v, da bo veljalo:

$$U(u,v) = 0$$

$$V(u,v) = 0$$

Parametra u, v določimo numerično, npr. z uporabo fsolve v Matlabu.

Ko imamo ustrezna u, v, izračunamo:

$$\Delta x_i \overset{(**)}{=} \frac{l_i}{\sqrt{1+(\cdots)^2}}$$

$$\Delta y_i \overset{(*)}{=} \left(v - u\sum \mu_i\right) \cdot \Delta x_i$$

$$x_i \overset{(***)}{=} x_0 + \sum \Delta x_j$$

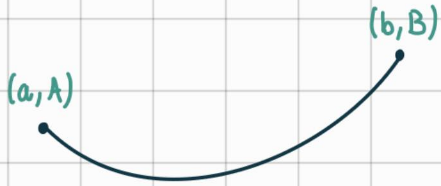
$$y_i = y_0 + \sum \Delta y_j$$

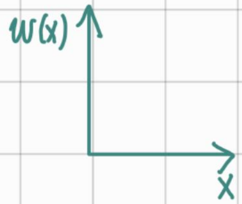
Kako izbrati začetni približek u_0, v_0 za u, v?

- $u_0 < 0$ (ker velja $u = \frac{1}{-2\lambda_i \Delta x_i}$ in pričakujemo $\lambda_i > 0$ za minimizacijo)
- $v_0 < 0$ (razen ko je desno obesišče precej višje od levega)

ZVEZNA VERIŽNICA

Tanka upogljiva homogena vrv dolžine l je vpeta v obesišči (a,A) in (b,B).





Zanima nas oblika verižnice $w(x)$.

Iščemo minimum energijskega funkcionala:

$$\Phi(w) = \int_a^b w(x) \cdot \underbrace{\sqrt{1+w'(x)^2}\, dx}_{ds = \sqrt{dx^2+dy^2}}$$

Pri pogoju:

$$\int_a^b \sqrt{1+w'(x)^2}\, dx = l$$

Za dani variacijski račun bi morali uporabiti Euler-Lagrangeevo enačbo:

$$\frac{\partial L}{\partial w} - \frac{d}{dx}\left(\frac{\partial L}{\partial w'}\right) = 0$$

$$L = w\sqrt{1+w'^2} - \underbrace{\lambda\sqrt{1+w'^2}}_{\text{vez}} = L(x, w, w')$$

Če je L neodvisen od x, se enačba poenostavi v Beltramijevo identiteto:

$$L - w' \cdot \frac{\partial L}{\partial w'} \equiv C$$

Naš izraz vstavimo v identiteto in dobimo:

$$w\sqrt{1+w'^2} - \lambda \cdot \sqrt{1+w'^2} - w' \cdot \left(\frac{w w'}{\sqrt{1+w'^2}} - \frac{\lambda w'}{\sqrt{1+w'^2}}\right) \equiv C$$

$$\frac{w(1+\cancel{w'^2}) - \lambda(1+\cancel{w'^2}) - \cancel{w}w'^2 - \cancel{\lambda}w'^2}{\sqrt{1+w'^2}} \equiv C$$

$w - \lambda = C \cdot \sqrt{1 + w'^2}$ (nelin. dif. enačba 1. reda)

Nastavek: $w' = \sinh(p)$

$w - \lambda = C \cdot \sqrt{1 + \sinh^2 p} \overset{\cosh^2 - \sinh^2 = 1}{=} C \cdot \cosh p \quad / \frac{d}{dx}$

$\frac{dw}{dx} = w' = \underline{\sinh p = C \cdot \sinh p \cdot \frac{dp}{dx}}$

$dx = C \cdot dp$

$x = Cp + D$

$p = \frac{x - D}{C}$

$\Rightarrow w(x) = \lambda + \underset{\underset{0}{\vee}}{C} \cdot \cosh \frac{x - D}{C}$

Določiti moramo λ, C, D, da zadostimo robnim pogojem (levo in desno obesišče ter dolžina vrvi):

i) $A = \lambda + C \cdot \cosh(\underbrace{\tfrac{a - D}{C}}_{=:u})$

ii) $B = \lambda \cdot C \cdot \cosh(\underbrace{\tfrac{b - D}{C}}_{=:v})$

iii) $l = \int_a^b \sqrt{1 + w'^2}\, dx = \int_a^b \cosh(\frac{x - D}{C})\, dx = C \cdot (\sinh(\underbrace{\tfrac{b - D}{C}}_{=:v}) - \sin(\underbrace{\tfrac{a - D}{C}}_{=:u}))$

$/ : C$

i) $\frac{A}{C} = \frac{\lambda}{C} + \cosh u$

ii) $\frac{B}{C} = \frac{\lambda}{C} + \cosh v$

iii) $\frac{l}{C} = \sinh v - \sinh u$

$$v-u=\frac{b-a}{c}>0 \Rightarrow \frac{1}{v-u}=\frac{c}{b-a}$$

$$\frac{(ii)-(i)}{v-u}: \quad \frac{B-A}{c(v-u)}=\frac{B-A}{b-a}=\frac{\cosh v-\cosh u}{v-u}\overset{\text{adicijski izrek}}{=}\frac{\sinh\frac{v+u}{2}\cdot\sinh\frac{v-u}{2}}{\frac{v-u}{2}} \qquad (iv)$$

$$\frac{(iii)}{v-u}: \quad \frac{l}{b-a}=\frac{\sinh v-\sinh u}{v-u}\overset{\text{adicijski izrek}}{=}\frac{\cosh\frac{v+u}{2}\cdot\sinh\frac{v-u}{2}}{\frac{v-u}{2}} \qquad (v)$$

$$\frac{(iv)}{(v)}: \quad \frac{B-A}{l}=\tanh\frac{v+u}{2} \qquad (vi)$$

$$\sinh\frac{v+u}{2}=\sinh\frac{v+u}{2}\cdot\frac{1}{\sqrt{1}}$$

$$=\sinh\frac{v+u}{2}\cdot\frac{1}{\sqrt{\cosh^2(x)-\sinh^2(x)}}$$

$$=\tanh\frac{v+u}{2}\cdot\cancel{\cosh\frac{v+u}{2}}\cdot\frac{\frac{1}{\cosh\frac{u+v}{2}}}{\sqrt{1-\tanh^2\frac{u+v}{2}}}$$

$$\overset{(vi)}{=}\frac{B-A}{l}\cdot\frac{1}{\sqrt{1-\left(\frac{B-A}{l}\right)^2}}$$

$$(iv) \Rightarrow \sinh\frac{v-u}{2}=\frac{B-A}{b-a}\cdot\frac{l}{B-A}\cdot\sqrt{1-\left(\frac{B-A}{l}\right)^2}\cdot\frac{v-u}{2}$$

$$=\underbrace{\frac{l}{b-a}\cdot\sqrt{1-\left(\frac{B-A}{l}\right)^2}}_{\rho}\cdot\underbrace{\frac{v-u}{2}}_{z}$$

$$\Rightarrow \sinh z=\rho\cdot z$$

Navadna iteracija:

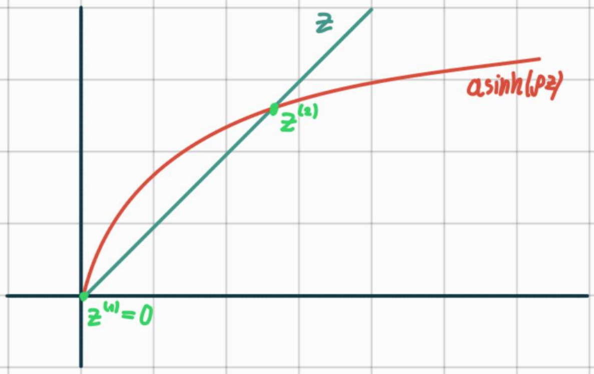
1) $\frac{\sinh z}{\rho}=z$ na žalost ne konvergira ☹

2) $\operatorname{asinh}(\rho z)=z$ deluje ☺

z_0 začetni približek

$z_{k+1} = \operatorname{asinh}(\rho z_k) \quad , \; k = 0, 1, \ldots$

Izkaže se, da ima enačba $\operatorname{asinh}(\rho z) = z$ natanko dve rešitvi, če je $\rho > 1$:



Kaj pa pomeni $\rho > 1$?

$\rho > 1$

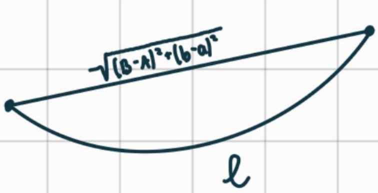
$$\frac{\ell}{b-a}\sqrt{1-\left(\frac{B-A}{2}\right)^2} > 1$$

$$\frac{\ell^2}{(b-a)^2}\cdot\left(1-\left(\frac{B-A}{2}\right)^2\right) > 1$$

$$\ell^2 - (B-A)^2 > (b-a)^2$$

$$\ell^2 > (B-A)^2 + (b-a)^2$$

To pomeni, da vrv ni prekratka.



Pokažimo, da iteracija konvergira k $z^{(2)}$, če je $z_0 > 0$:

$z^{(k)} > z_k \Rightarrow z_{k+1} > z_k$

$\underline{\operatorname{asinh} z_k > z_k}$

Očitno iz slike

$\underline{z^{(k)} < z_k \Rightarrow z_{k+1} < z_k}$

$\underline{\operatorname{asinh} z_k < z_k}$

Očitno iz slike

Ko določimo $z := \lim_{k \to \infty} z_k$, iz $z = \frac{v-u}{2}$ in $\operatorname{atanh} \frac{B-A}{\ell} = \frac{v+u}{2}$ dobimo:

$v = \operatorname{atanh} \frac{B-A}{\ell} + z$

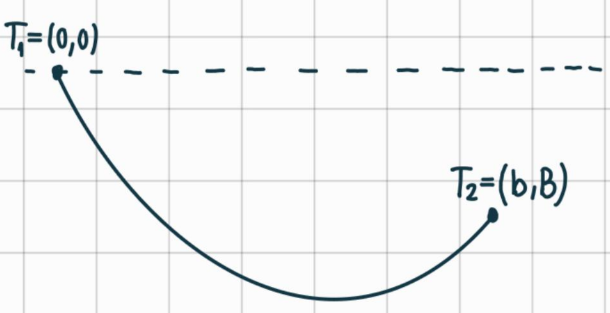
$u = \operatorname{atanh} \frac{B-A}{\ell} - z$

$C = \frac{b-a}{v-u}$

$D = b - cv = \ldots = \frac{av - bu}{v-u}$

$\lambda = A - C \cdot \cosh \frac{a-D}{c}$

BRAHISTOKRONA



$$X(\theta) = \frac{k^2}{2}(\theta - \sin\theta)$$
$$Y(\theta) = -\frac{k^2}{2}(1 - \cos\theta)$$

$\theta \in [0, \theta^*]$

θ^* rešitev $g(\theta) = 1 - \cos\theta + \frac{B}{b}(\theta - \sin\theta) = 0$

$K = \sqrt{\frac{2b}{\theta^* - \sin\theta^*}}$

Čas potovanja po krivulji:

$$T = \int_0^b \sqrt{\frac{1 + \big(\overset{=\frac{dy}{dx}}{y'(x)}\big)^2}{-2g\,y(x)}}\, dx \qquad \text{(splošna formula)}$$

Brahistokrona:

$$dx = \frac{K^2}{2}(1 - \cos\theta)\, d\theta$$
$$dy = -\frac{K^2}{2} \sin\theta \, d\theta$$

$$\Rightarrow T = \int_0^{\theta^*} \sqrt{\frac{1 + \left(\frac{-K^2/2 \cdot \sin\theta}{K^2/2 \cdot (1-\cos\theta)}\right)}{-2g \cdot (-K^2/2 \cdot (1 - \cos\theta))}} \cdot \frac{K^2}{2}(1 - \cos\theta)\, d\theta$$

$$= \int_0^{\theta^*} \sqrt{\frac{1 + \frac{\sin^2\theta}{(1-\cos\theta)^2}}{g K^2 (1 - \cos\theta)}} \cdot \frac{K^2}{2}(1 - \cos\theta)\, d\theta$$

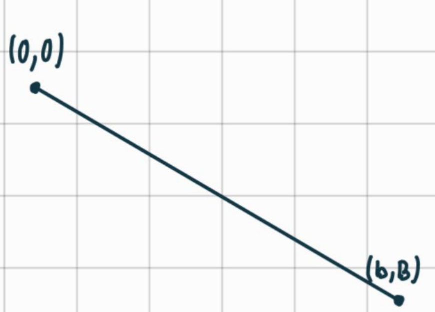
$$= \int_0^{\theta^*} \sqrt{\frac{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}{g K^2 (1 - \cos\theta)^2}} \cdot \frac{K^2}{2}(1 - \cos\theta)\, d\theta$$

$$= \int_0^{\theta^*} \sqrt{\frac{2(1 - \cos\theta)}{g K^2 (1 - \cos\theta)^3}} \cdot \frac{K^2}{2}(1 - \cos\theta)\, d\theta$$

$$= \int_0^{\theta^*} \frac{K}{\sqrt{2g}}\, d\theta$$

$$= \frac{K}{\sqrt{2g}} \cdot \theta^*$$

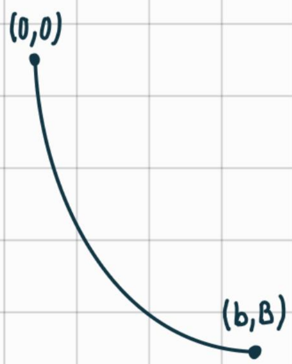
DALJICA



$$y = \frac{B}{b} x$$
$$y' = \frac{B}{b}$$

$$T = \int_0^b \sqrt{\frac{1+\left(\frac{B}{b}\right)^2}{-2g\frac{B}{b}x}}\, dx$$
$$= \int_0^b \sqrt{\frac{b^2+B^2}{-2gBbx}}\, dx$$
$$= \sqrt{\frac{b^2+B^2}{-2gBb}} \int_0^b x^{-1/2}\, dx$$
$$= \sqrt{\frac{b^2+B^2}{-2gBb}} \cdot \frac{b^{1/2}}{1/2}$$
$$= \sqrt{\frac{2(b^2+B^2)}{-gB}}$$

PARABOLA



$$y = a(x-b)^2 + c$$
$$y' = 2a(x-b)$$

$$y(b) = B \Rightarrow c = B$$
$$y(0) = 0 \Rightarrow ab^2 + B = 0 \Rightarrow a = -\frac{B}{b^2}$$

$$\Rightarrow y(x) = -\frac{B}{b^2}(x-b)^2 + B = -\frac{B}{b^2}x^2 + \frac{B}{b}2x$$

INVERZNI PROBLEM BRAHISTOKRONE

Za izbrano krivuljo želimo poiskati potencialno polje, da bo naša krivulja tista, kjer je čas potovanja najkrajši.

Naj bo polje $U(y)$ odvisno samo od višine.

$\frac{1}{2}mv^2 = -U(y)$ ... ohranitev energije

$v_0 = 0$
$U_0 = 0$

$$v = \sqrt{-\frac{2}{m}U(y)}$$

Čas potovanja: $T = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{-\frac{2}{m}U}}\,dx$

Beltramijeva identiteta: $L - y' \cdot \frac{\partial L}{\partial y'} \equiv C$

$$\Rightarrow \sqrt{\frac{1+y'^2}{-2/m \cdot U}} - y' \cdot \frac{y'}{\sqrt{-2/m \cdot U} \cdot \sqrt{1+y'^2}} \equiv C \quad / \cdot \sqrt{-\frac{2}{m} \cdot U} \cdot \sqrt{1+y'^2}$$

$$1 + y'^2 - y'^2 \equiv C \cdot \sqrt{-\frac{2}{m} \cdot U} \cdot \sqrt{1+y'^2}$$

$$1 \equiv -\underbrace{C^2 \cdot \frac{2}{m}}_{=A} \cdot U \cdot (1+y'^2)$$

$$U = -\frac{A^{-1}}{1+y'^2} \qquad A^{-1} =: \tilde{A} > 0$$

Cikloida:

$$x(\theta) = D(\theta - \sin\theta)$$
$$y(\theta) = -D(1-\cos\theta) \qquad (*)$$

$$y'(\theta) = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-\sin\theta}{1-\cos\theta}$$

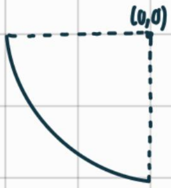
$$\Rightarrow U = -\frac{\tilde{A}}{1+\frac{\sin^2\theta}{(1-\cos\theta)^2}} = -\frac{\tilde{A}(1-\cos\theta)^2}{1-2\cos\theta+\underbrace{\cos^2\theta+\sin^2\theta}_{1}} = -\frac{\tilde{A}(1-\cos\theta)^2}{2(1-\cos\theta)} \overset{(*)}{=} -\frac{\tilde{A}}{2}\cdot\frac{y}{-D} = \underset{\substack{\vee \\ 0}}{B} \cdot y$$

$\Rightarrow$ Sila: $F(y) = -U'(y) = -B$

Krožnica:

$$x^2 + y^2 = R^2 \,, \quad x, y \leq 0$$

$$\frac{d}{dx}: \; 2x + 2yy' = 0 \quad \Rightarrow \quad y' = -\frac{x}{y}$$



$$1 + y'^2 = 1 + \frac{x^2}{y^2} = \frac{R^2}{y^2}$$

$$\Rightarrow U = -\frac{\tilde{A} y^2}{R^2} = -\underset{\substack{\vee \\ 0}}{B}\, y^2$$

$$\Rightarrow \text{Sila: } F(y) = -U'(y) = 2B\underset{\substack{\wedge \\ 0}}{y}$$