Šibki odvod:

$I \subseteq \mathbb{R}$

$L^1_{loc}(I) := \{u : I \to \mathbb{R} \; ; \; \forall p \in I : \exists \varepsilon > 0 : \int_{p-\varepsilon}^{p+\varepsilon} |u(y)| dy < \infty\}$

$u \in L^1_{loc}(I)$

$v \in L^1_{loc}(I)$ šibki odvod $:\Leftrightarrow \forall \varphi \in C_c^\infty(I) : \int_I u(x)\varphi'(x)\,dx = -\int_I v(x)\varphi(x)\,dx$

$C_c^\infty(I) := \{\varphi \in C^\infty(I) \; ; \; \overline{\{\varphi \neq 0\}}^{komp} \subseteq I\}$

Sobolijev prostor:

$1 \leq p \leq \infty$

$W^{k,p}(I) := \{u \in L^p(I) \; ;$ obstaja šibki odvod $u^{(j)} \in L^p(I)$ za vse $0 \leq j \leq k\}$

Norma na $W^{k,p}(I)$:

$\|u\|_{W^{k,p}} := \left(\sum_{j=0}^{k} \|u^{(j)}\|^p_{L^p}\right)^{1/p}$

$\|u\|_{W^{k,\infty}} := \max_{0 \leq j \leq k} \{\|u^{(j)}\|_{L^\infty}\}$

Kjer je:

$\|u\|_{L^p} := \left(\int_I |u(x)|^p dx\right)^{1/p}$

$$\|u\|_{L^\infty} := \operatorname*{ess\,sup}_{x\in I} |u(x)|$$

1) $I=(-1,1)$

$u(x)=|x|$

a) Izračunajte šibki odvod u.

$$\int_I u(x)\varphi'(x)\,dx = \int_{-1}^{1}|x|\varphi'(x)\,dx = \int_0^1 x\varphi'(x)\,dx - \int_{-1}^{0} x\varphi'(x)\,dx$$

per partes

$$= x\varphi(x)\Big|_0^1 - \int_0^1 \varphi(x)\,dx - x\varphi(x)\Big|_{-1}^{0} + \int_{-1}^{0}\varphi(x)\,dx$$

$$= \underbrace{\underbrace{\varphi(1)}_{=0} - \underbrace{\varphi(-1)}_{=0}}_{\varphi\in C_c^\infty(-1,1)} - \int_0^1\varphi(x)\,dx + \int_{-1}^{0}\varphi(x)\,dx$$

$$= \int_0^1 (-1)\varphi(x)\,dx + \int_{-1}^{0} 1\cdot\varphi(x)\,dx$$

$$\Rightarrow v(x) = \begin{cases} 1 \; ; & x\in(0,1) \\ -1 \; ; & x\in(-1,0)\end{cases}$$

b) $u_\varepsilon(x) = \sqrt{x^2+\varepsilon^2} \in C^\infty(\bar{I})$ za $\varepsilon>0$

Pokažite: $u_\varepsilon \xrightarrow{\varepsilon\downarrow 0} u$ v $W^{1,p}$ za $1\le p<\infty$

$$\Leftrightarrow \|u_\varepsilon - u\|_{W^{1,p}} \xrightarrow{\varepsilon\downarrow 0} 0$$

$$\Leftrightarrow \|u_\varepsilon - u\|_{W^{1,p}}^p = \|u_\varepsilon - u\|_{L^p}^p + \|(u_\varepsilon - u)'\|_{L^p}^p \xrightarrow{\varepsilon\downarrow 0} 0$$

$$\Leftrightarrow \|u_\varepsilon - u\|_{L^p}^p \longrightarrow 0 \text{ in } \|(u_\varepsilon - u)'\|_{L^p}^p \longrightarrow 0$$

Velja: $|f_n|\le M$, $f_n \longrightarrow f$ enakomerno $\Rightarrow \int |f_n - f|^p\,dx \longrightarrow 0$

$$\|u_\varepsilon - u\|_{L^p}^p = \int_{-1}^{1} \left| \sqrt{x^2+\varepsilon^2} - |x| \right|^p dx \longrightarrow 0$$

$$\|u_\varepsilon' - u'\|_{L^p}^p = \int_{-1}^{1} \Big| \underbrace{\frac{x}{\sqrt{x^2+\varepsilon^2}}}_{\substack{\downarrow\, \varepsilon\to 0,\ x\neq 0 \\ \frac{x}{|x|}}} - \underbrace{v(x)}_{\substack{\| \, x\neq 0 \\ \frac{x}{|x|}}} \Big|^p dx \longrightarrow 0$$

Opomba: $\overline{C^\infty(\overline{I})} = W^{k,p}(I)$, $p \in [1,\infty)$

c) Pokažite: u ni limita C^1 funkcij v prostoru $W^{1,\infty}(I)$

$$\lim_{i\to\infty} v_i = u \iff \lim_{i\to\infty} \|v_i - u\|_{W^{1,p}} = 0$$

$$\iff \lim_{i\to\infty} \max\{\sup_I |u - v_i|, \sup_I |u' - v_i'|\} = 0$$

$$\Rightarrow \sup_I |u - v_i| \to 0 \text{ in } \sup_I |u' - v_i'| \to 0$$

Limita v_i' je zvezna, ker so v_i zvezne

29.5.

$$\overline{C^\infty} = W^{k,p}, \quad 1 < p < \infty$$

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2) $I = (0,2)$

$$a(x) = \begin{cases} 1; & 0 < x < 1 \\ 2; & 1 < x < 2 \end{cases}$$

Šibki robni problem: $-(a(x)\,u'(x))' = 1$ na I, $u(0) = u(2) = 0$

Določite šibko rešitev $u \in W^{2,2}(I)$.

Želimo: $-\int_0^2 (a(x)\,u'(x))\,\varphi'(x)\,dx = \int_0^2 1 \cdot \varphi(x)\ \ \forall \varphi \in C_c^\infty(I)$ ali $C^\infty(\overline{I})$

$$-\int_0^2 (a(x)u'(x))\,\varphi'(x)\,dx \overset{\text{per partes}}{=} -a(x)u'(x)\varphi(x)\Big|_0^2 + \int_0^2 a(x)u'(x)\varphi'(x)\,dx$$

$$= -a(2)u'(2)\varphi(2) + a(0)u'(0)\varphi(0)$$

$$+\int_0^1 a(x)u'(x)\varphi'(x)\,dx + \int_1^2 a(x)u'(x)\varphi'(x)\,dx$$

$$= -a(2)u'(2)\varphi(2) + a(0)u'(0)\varphi(0)$$

$$+\int_0^1 1\cdot u'(x)\varphi'(x)\,dx + \int_1^2 2\cdot u'(x)\varphi'(x)\,dx$$

$$\overset{\text{per partes}}{=} a(0)u'(0)\varphi(0) - a(2)u'(2)\varphi(2)$$

$$-\int_0^1 u''(x)\varphi(x)\,dx + u'(x)\varphi(x)\Big|_0^1$$

$$-2\int_1^2 u''(x)\varphi(x)\,dx + 2u'(x)\varphi(x)\Big|_1^2$$

$$= \overset{:=1}{a(0)}\,\cancel{u'(0)}\,\varphi(0) - \overset{:=2}{a(2)}\,\cancel{u'(2)}\,\varphi(2)$$

$$+u'(1^-)\varphi(1^-) - \cancel{u'(0)}\varphi(0)$$

$$+2\cancel{u'(2)}\varphi(2) - 2u'(1^+)\varphi(1^+)$$

$$-\int_0^1 u''(x)\varphi(x)\,dx - 2\int_1^2 u''(x)\varphi(x)\,dx$$

$$= \underbrace{(u'(1^-) - 2u'(1^+))}_{\substack{:=0 \\ (\text{želimo } C^1)}}\varphi(1) - \int_0^2 v(x)\varphi(x) = \int_0^2 f(x)$$

$$v(x) := \begin{cases} u''(x) & ; \; x \in (0,1) \\ 2u''(x) & ; \; x \in (1,2) \end{cases}$$

$$\Rightarrow u''(x) = \begin{cases} -1 & ; \; x \in (0,1) \\ -\frac{1}{2} & ; \; x \in (1,2) \end{cases}$$

$$\Rightarrow u'(x) = \begin{cases} -x + A & ; \quad x \in (0,1) \\ -\frac{x}{2} + C & ; \quad x \in (1,2) \end{cases}$$

$$\Rightarrow u(x) = \begin{cases} -\frac{x^2}{2} + Ax + B & ; \quad x \in (0,1) \\ -\frac{x^2}{4} + Cx + D & ; \quad x \in (1,2) \end{cases}$$

Pogoji:

- $u(0) = u(2) = 0$
- $u'(1^-) = 2u'(1^+)$
- $u(1^-) = u(1^+)$

$u(0) = B = 0$

$u(2) = -1 + 2C + D = 0$

$$\begin{aligned} u'(1^-) &= -1 + A \\ &\parallel \\ 2u'(1^+) &= -1 + 2C \end{aligned}$$

$\Rightarrow A = 2C$

$$\begin{aligned} u(1^-) &= -\tfrac{1}{2} + A \\ &\parallel \\ u(1^+) &= -\tfrac{1}{4} + C + D \end{aligned}$$

$\Rightarrow -\frac{1}{2} + 2C = -\frac{1}{4} + C + \overbrace{1 - 2C}^{D}$

$\Rightarrow -\frac{1}{2} + 2C = \frac{3}{4} - C$

$\Rightarrow 3C = \frac{5}{4}$

$\Rightarrow C = \frac{5}{12}$

$\Rightarrow A = \frac{5}{6}$

$\Rightarrow D = 1 - 2C = 1 - \frac{5}{6} = \frac{1}{6}$

$$\Rightarrow u(x) = \begin{cases} -\frac{x^2}{2} + \frac{5}{6}x & ; x \in (0,1) \\ -\frac{x^2}{4} + \frac{5}{12}x + \frac{1}{6} & ; x \in (1,2) \end{cases}$$

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3) $I = (-1,1)$
$u(x) = x|x|$

a) Pokažite $u \in C^1(\overline{I})$ in določite u'.

$$u(x) = \begin{cases} x^2 & ; x \geq 0 \\ -x^2 & ; x < 0 \end{cases}$$

$$u'(x) = \begin{cases} 2x & ; x > 0 \\ -2x & ; x < 0 \end{cases}$$

$$u'(0) = \lim_{h \to 0} \frac{u(0+h) - u(0)}{h} = \lim_{h \to 0} \frac{h|h| - 0}{h} = \lim_{h \to 0} |h| = 0$$

$$\lim_{x \to 0} u'(x) = \lim_{x \to 0} \pm 2x = 0 = u'(0)$$

$\checkmark$

b) Izračunajte drugi šibki odvod u''. Za katere $p \in [1, \infty]$ velja $u \in W^{2,p}(I)$?

$$\int_I u'(x) \varphi'(x)\, dx = -\int_I v(x) \varphi(x)\, dx$$

$$\int_{-1}^{1} u'(x)\varphi'(x)\,dx = \int_{-1}^{0} -2x\,\varphi'(x)\,dx + \int_{0}^{1} 2x\,\varphi'(x)\,dx$$

$$= -2\left[x\varphi(x)\Big|_{-1}^{0} - \int_{-1}^{0}\varphi(x)\,dx\right] + 2\left[x\varphi(x)\Big|_{0}^{1} - \int_{0}^{1}\varphi(x)\,dx\right]$$

$\varphi \in C_c^\infty(-1,1)$
$\Rightarrow \varphi(-1)=\varphi(1)=0$

$$= 2\int_{-1}^{0}\varphi(x)\,dx - 2\int_{0}^{1}\varphi(x)\,dx$$

$$= -\int_{-1}^{1} v(x)\,\varphi(x)\,dx$$

$$\Rightarrow v(x) = \begin{cases} 2 & ; \ x>0 \\ -2 & ; \ x<0 \end{cases}$$

$u \in W^{2,p}(I)$?

$u(x) = x|x|$

$$u'(x) = \begin{cases} 2x & ; \ x \in (0,1) \\ -2x & ; \ x \in (-1,0) \end{cases}$$

$$u''(x) = \begin{cases} 2 & ; \ x \in (0,1) \\ -2 & ; \ x \in (-1,0) \end{cases}$$

Vse to je omejeno na I, torej $u \in W^{2,p}(I)$ za vse $p \in [1,\infty]$.

c) Pokažite, da je u šibka rešitev problema $u''(x) = 2\,\mathrm{sgn}(x)$ na I, $u(-1)=-1$, $u(1)=1$.

$$\int_{-1}^{1} u''(x)\varphi(x)\,dx = \int_{-1}^{1} 2\,\mathrm{sgn}(x)\varphi(x)\,dx$$

$$\Rightarrow u''(x) = 2\,\mathrm{sgn}(x)$$

Robni pogoji veljajo ☺