Harmonične funkcije:

$u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, $u \in C^2(\Omega)$

$$\Delta u = \sum_{i=1}^{n} \partial^2_{x_i x_i} u = 0$$

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1) Dokaži: Laplaceova enačba invariantna za translacije in rotacije

Natančneje: $\Delta u(x) = 0$, $b \in \mathbb{R}^n$, $M \in O(n) \Rightarrow \Delta u(Mx+b) = 0$

$$\Delta(u(Mx+b)) = \sum_{i=1}^{n} \partial_{x_i^2}(u(Mx+b))$$

$$= \sum_{i=1}^{n} \partial_{x_i^2} u\left(\left[\sum_{k=1}^{n} m_{jk} x_k + b_j\right]_j\right)$$

$$\partial_{x_i} u(f_1(x), \dots, f_n(x)) = u_{x_i} \cdot \frac{\partial f_1}{\partial x_i} + \dots + u_{x_i} \cdot \frac{\partial f_n}{\partial x_i}$$

$$= \sum_{i=1}^{n} \partial_{x_i}\left(\sum_{j=1}^{n} u_{x_j}\left(\left[\sum_{k=1}^{n} m_{jk} x_k + b_j\right]_j\right) m_{ji}\right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\ell=1}^{n} u_{x_j x_\ell}(Mx+b)\, m_{ji}\, m_{\ell i}$$

$$\overset{M^T M = I_n}{=} \sum_{j=1}^{n} \sum_{\ell=1}^{n} u_{x_j x_\ell}(Mx+b)\, \delta_{j\ell}$$

$$= \sum_{j=1}^{n} u_{x_j x_j}(Mx+b)$$

$$= (\Delta u)(Mx+b)$$

$$= 0$$

2) Homogeni harmonični polinom stopnje n:

$$P(x,y)=\sum_{i=0}^{n} a_i x^i y^{n-i}\quad,\ \text{kjer } \Delta P=0$$

V_n = prostor homogenih harmoničnih polinomomov stopnje n

$\dim V_n = ?$

$$\Delta P = \partial_{x^2}\sum_{i=0}^{n} a_i x^i y^{n-i} + \partial_{y^2}\sum_{i=0}^{n} a_i x^i y^{n-i}$$

$$= \partial_x \sum_{i=1}^{n} i a_i x^{i-1} y^{n-i} + \partial_y \sum_{i=0}^{n-1} (n-i) a_i x^i y^{n-i-1}$$

$$= \sum_{i=2}^{n} i(i-1) x^{i-2} y^{n-i} + \sum_{i=0}^{n-2} (n-i)(n-i-1) a_i x^i y^{n-i-2} = *$$

$V_0 = \{\text{konstante}\} \Rightarrow \dim V_0 = 1$

$V_1 = \{a_0 x + a_1 y\} \Rightarrow \dim V_1 = 2$

$$* = \sum_{i=0}^{n-2} (i+1)(i+2) a_{i+2} x^i y^{n-i-2} + \sum_{i=0}^{n-2} (n-i)(n-i-1) a_i x^i y^{n-i-2}$$

$$= \sum_{i=0}^{n-2} \left[(i+1)(i+2) a_{i+2} + (n-i)(n-i-1) a_i\right] x^i y^{n-i-2}$$

$n \geq 2$: $(i+1)(i+2) a_{i+2} + (n-i)(n-i-1) a_i = 0$

$$a_{i+2} = -\frac{(n-i)(n-i-1)}{(i+1)(i+2)} \cdot a_i$$

$\Rightarrow a_0$ in a_1 določata a_i, $i \geq 2$

$\Rightarrow \dim V_n = 2$

Opomba: P splošni polinom

$$\Rightarrow P = P_0 + P_1 + \ldots + P_n, \quad P_i \text{ homogeni}$$

$$\Delta P = \Delta P_0 + \Delta P_1 + \ldots + \Delta P_n = 0 \iff \forall i: \Delta P_i = 0$$

$$D = \{(x,y) \in \mathbb{R}^2 ; x^2+y^2=1\}$$

$\Delta u = 0$ na D
$u = f$ na ∂D, f zvezna

Poissonovo jedro: $P_r(t) = \frac{1-r^2}{1-2r\cos(2\pi t)+r^2}$

$$\Rightarrow u(r\cos(2\pi t), r\sin(2\pi t)) = (P_t * f)(t) = \int_0^1 f(\tau) P_r(t-\tau)\, d\tau$$

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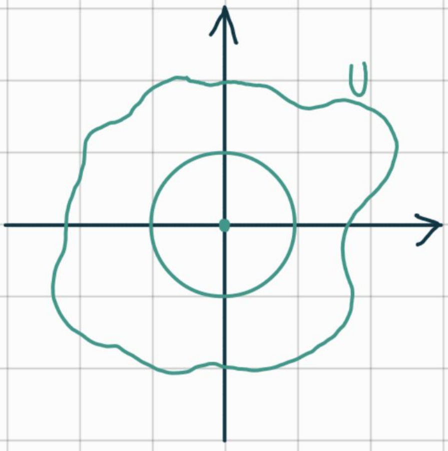
3) $U \subseteq \mathbb{R}^2$ odprta
$a \in U$
$u \in C^2(U \setminus \{a\})$ omejena harmonična

Dokažite: u lahko razširimo do harmonične na U



$$\exists \varepsilon > 0 : D := K(a, \varepsilon) \subseteq U$$

Rešujemo:

$$\Delta v = 0 \text{ na } D$$
$$v = u \text{ na } \partial D$$

$\Rightarrow$ Dobimo enolično rešitev v in hočemo razširiti $u(a) = v(a)$

$w := u - v$

$$\Delta w = 0 \text{ na } D \setminus \{a\}$$
$$w = 0 \text{ na } \partial D$$

$D_\delta := D \setminus \overline{K(a, \delta)}$

w zvezna na $\overline{D_\delta}$ in $\Delta w = 0$ na ∂D_δ

$\Rightarrow$ Po principu maksima ima w max na $S(a, \varepsilon)$ ali $S(a, \delta)$

$\Rightarrow w \leq \max\{0, \max\limits_{S(a,\delta)} w\}$

Ampak ne vemo še, kaj je $\max\limits_{S(a,\delta)} w$ ...

$\log\left(\frac{|\vec{x} - \vec{a}|^2}{\varepsilon}\right)$ zadošča zgornjemu problemu

$w_\gamma := w + \gamma \cdot \log\left(\frac{|\vec{x} - \vec{a}|^2}{\varepsilon}\right)$ tudi zadošča

u, v omejeni $\Rightarrow w$ omejena

i) $\gamma > 0$:

Princip maksima: $w_\gamma \leq \max\{0, \max\limits_{S(a,\delta)} w_\gamma\}$

$\max\limits_{S(a,\sigma)} w_\gamma \leq 0$ za dovolj majhen σ

$\Rightarrow w_\gamma \leq 0$ na vseh D_σ za dovolj majhne σ

$\overset{\sigma \downarrow 0}{\Rightarrow} w_\gamma \leq 0$ na $D \setminus \{a\}$ za vse $\gamma > 0$

$\overset{\gamma \downarrow 0}{\Rightarrow} w \leq 0$ na $D \setminus \{a\}$

ii) $\gamma < 0$:

Princip minima: $w_\gamma \geq \min\{0, \min\limits_{S(a,\sigma)} w_\gamma\}$

$\min\limits_{S(a,\sigma)} w_\gamma \geq 0$ za dovolj majhen γ

Enako kot zgoraj ...

$\Rightarrow w \geq 0$ na $D \setminus \{a\}$

$\Rightarrow w = 0$ na $D \setminus \{a\}$

15.5.

Naj bo $P_r(t) = \frac{1-r^2}{1-2r\cos(2\pi t)+r^2}$ Poissonovo jedro.

Če je $f \in C^0(S^1)$, potem je $F(re^{2\pi i t}) = P_r * f = \int_0^1 f(e^{2\pi i \tau}) P_r(t-\tau)\, d\tau$ harmonična razširitev:

$\Delta F = 0$
$F|_{\partial D} = f$

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4) $u: \mathbb{D} \to \mathbb{R}$ nenegativna harmonična na enotskem disku in se zvezno razširi na $\partial \mathbb{D}$

$p \in \mathbb{D}$

Dokažite: $\frac{1-|p|}{1+|p|} \cdot u(0,0) \leq u(p) \leq \frac{1+|p|}{1-|p|} \cdot u(0,0)$

$f(t) = u(e^{2\pi i t})$

$$\begin{aligned} \Delta u &= 0 \\ u|_{\partial D} &= u|_{\partial D} \end{aligned} \overset{\text{enoličnost}}{\Longrightarrow} F = u$$

$$F(re^{2\pi i t}) = u(\underbrace{re^{2\pi i t}}_{p}) = \int_0^1 u(e^{2\pi i \tau}) \cdot \frac{1-r^2}{1-2r\cos(2\pi(t-\tau))+r^2}\, d\tau$$

$$= \int_0^1 u(e^{2\pi i \tau}) \cdot \underbrace{\frac{1-|p|^2}{1-2|p|\cos(2\pi(t-\tau))+|p|^2}}_{\substack{\geq\ \ \leq \\ C_1\ \ C_2}}\, d\tau.$$

$$\frac{1-|p|^2}{1+2|p|+|p|^2} \overset{\cos=-1}{\leq} \frac{1-|p|^2}{1-2|p|\cos(2\pi(t-\tau))+|p|^2} \overset{\cos=1}{\leq} \frac{1-|p|^2}{1-2|p|+|p|^2}$$

$$\frac{1-|p|^2}{1+2|p|+|p|^2} = \frac{(1-|p|)(1+|p|)}{(1+|p|)^2} = \frac{1-|p|}{1+|p|} = C_1$$

$$\frac{1-|p|^2}{1-2|p|+|p|^2} = \frac{(1-|p|)(1+|p|)}{(1-|p|)^2} = \frac{1+|p|}{1-|p|} = C_2$$

$$\Rightarrow u(p) \leq \int_0^1 u(e^{2\pi i \tau})\, C_1\, d\tau \overset{MVP}{=} C_1 \cdot u(0,0)$$

$$u(p) \geq \int_0^1 u(e^{2\pi i \tau})\, C_2\, d\tau \overset{MVP}{=} C_2 \cdot u(0,0)$$

Opomba: $1+|p|^2 - 2|p|\cos(2\pi(t-\tau)) = |p - e^{2\pi i \tau}|^2$

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5) $\mathbb{H} = \mathbb{R} \times (0,\infty)$

$u \in C^0(\overline{\mathbb{H}}) \cap C^2(\mathbb{H})$ navzgor omejena harmonična funkcija

$\exists M: u \leq M$

Dokažite: $\sup_{\overline{\mathbb{H}}} u = \sup_{\partial \mathbb{H}} u$

Naj bo $\Omega(r) = \{(x,y) \in \mathbb{H} \, ; \, x^2+y^2<r^2\}$.

Po principu maksima je maksimum za u dosežen na $\partial\Omega(r)$.

Ne vemo pa, na katerem delu roba je. Upamo, da je na x-osi, zato bomo "kaznovali" neskončnost. Kot včeraj bomo vzeli novo funkcijo $v_\varepsilon := u + \varepsilon\varphi$. Želimo, da za φ velja $\max\limits_{\partial\Omega(r)\setminus\partial\mathbb{H}} \varphi \xrightarrow{r\to+\infty} -\infty$ in $\Delta\varphi = 0$.

Potem bomo dobili:

$$\sup_{\Omega(r)} u \xleftarrow{\varepsilon\downarrow 0} \sup_{\Omega(r)} v_\varepsilon = \max_{\partial\Omega(r)} v_\varepsilon = \max_{\partial\mathbb{H}\cap[-r,r]} v_\varepsilon \leq \max u$$

Vzemimo:

$$\varphi = -\ln(x^2+(y-1)^2)$$

Dobimo:

$$\max_{\Omega(r)}(u-\varepsilon\ln(x^2+(y-1)^2)) = \max_{\partial\Omega(r)}(u-\varepsilon\ln(x^2+(y-1)^2))$$

$$= \max\left(\max_{\partial\Omega(r)_{(1)}} v_\varepsilon, \max_{\partial\Omega(r)_{(2)}} v_\varepsilon\right) \overset{\text{dovolj velik } r}{=} \max_{\partial\Omega(r)_{(2)}} v_\varepsilon$$

$$\Rightarrow \sup_{\mathbb{H}} v_\varepsilon = \sup_{\partial\mathbb{H}} v_\varepsilon \overset{\varepsilon>0}{\leq} \sup_{\partial\mathbb{H}} u$$

$$\begin{array}{l} \sup_{\mathbb{H}} v_\varepsilon \\ \quad \rotatebox{90}{$\geq$} \\ v_\varepsilon \xrightarrow{\varepsilon\downarrow 0} u \end{array}$$

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6) $\mathbb{H}$ zgornja polravnina

$$\Delta u = 0 \quad \text{na } \mathbb{H}$$
$$u = f \quad \text{na } \partial\mathbb{H}$$

f zvezna funkcija s kompaktnim nosilcem na $\partial\mathbb{H}$

Izpeljite reprezentacijsko funkcijo za rešitve problema

$\varphi: \mathbb{H} \to \mathbb{D}$ biholomorfna

$$\varphi(z) := \frac{z-i}{z+i}$$

Na disku: $v(re^{2\pi i t}) = \int_0^1 \tilde{f}(e^{2\pi i \tau}) P_r(t-\tau)\, d\tau = v(\varphi(z)) = u(z)$,

če je $\varphi(z) = re^{2\pi i t}$

$$\frac{z-i}{z+i} = re^{2\pi i t}$$

$$r = \left|\frac{z-i}{z+i}\right| = \frac{\sqrt{x^2+(y-1)^2}}{\sqrt{x^2+(y+1)^2}}$$

$$1-r^2 = 1 - \frac{x^2+(y-1)^2}{x^2+(y+1)^2} = \frac{(y+1)^2-(y-1)^2}{x^2+(y+1)^2} = \frac{4y}{x^2+(y+1)^2}$$

$$1-2r\cos(2\pi(t-\tau)) + r^2 = |\rho - \underbrace{e^{2\pi i(t-\tau)}}_{\varphi(s,0)}| = \left|\frac{z-i}{z+i} - \frac{s-i}{s+i}\right|^2 = \dots = 4\cdot\frac{(x-s)^2+y^2}{(1+s^2)(x^2+(y+1)^2)}$$

$$\left(e^{2\pi i \tau} = \frac{s-i}{s+i}\right)$$

$$2\pi i\, e^{2\pi i \tau}\, d\tau = \frac{s+i-(s-i)}{(s+i)^2}\, ds = 2\pi i \cdot \frac{s-i}{s+i}\, d\tau = \frac{2i}{(s+i)^2}\, ds$$

$$d\tau = \frac{1}{\pi(s+i)(s-i)}\, ds = \frac{1}{\pi(s^2+1)}\, ds$$

Tukaj je nekaj napak ...

Popravljeno:

Vemo: $v(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-r\cos(\theta t)+r^2} g(e^{it})\,dt$ rešitev za:

$$\Delta v = 0 \text{ na } D$$
$$v = f \text{ na } \partial D$$

Ideja: Kompozitum z $H \to D$, $w \mapsto \frac{w-i}{w+i} = p$

$$1-|p|^2 \overset{w=x+iy}{=} \dots = \frac{4y}{x^2+(y+1)^2}$$

$e^{it} = \frac{s-i}{s+i} \in \partial D$, $s \in \mathbb{R}$

$\Rightarrow dt = \frac{2}{1-s^2}\,ds$

$$\underline{1-2r\cos(\theta t)+r^2 = |e^{it}-p|^2}$$

$$|e^{it}-p|^2 = (e^{it}-p)(e^{-it}-\overline{p})$$
$$= 1+|p|^2 - e^{it}\overline{p} - pe^{-it}$$
$$= 1+|p|^2 - 2\operatorname{Re}(pe^{-it})$$
$$= 1+|p|^2 - 2\operatorname{Re}(re^{i(\theta-t)})$$

Računamo:

$$|e^{it}-p|^2 = \left|\frac{s-i}{s-i} - \frac{w-i}{w+i}\right|^2 = \dots = 4\cdot\frac{(x-s)^2+y^2}{(1+s^2)(x^2+(y+1)^2)}$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|p|^2}{|e^{it}-p|^2} v(e^{it})\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(s,0)\cdot\frac{4y}{x^2+(y+1)^2}\cdot\frac{1}{4}\frac{(1+s^2)(x^2+(y+1)^2)}{(x-s)^2+y^2}\cdot\frac{2}{1+s^2}\,ds$$

$$= \frac{y}{\pi} \int_{-\infty}^{\infty} f(s) \cdot \frac{1}{(x-s)^2 + y^2} \, ds$$

21.5.

1) $u : \mathbb{D} \cap \mathbb{H} \to \mathbb{R}$ harmonična
$u \in C^0(\overline{\mathbb{D} \cap \mathbb{H}})$
$u(x,0) = 0 \quad \forall x$

Dokažite: $v(x,y) = \begin{cases} u(x,y) & ; \ y \geq 0 \\ -u(x,y) & ; \ y < 0 \end{cases}$ harmonična na $\mathbb{D}$

Oznaka: $\fint_B f \, ds = \frac{1}{|B|} \int_B f \, ds$ (povprečje)

LPV: $\exists r_0 : \forall r \leq r_0 : \fint_{\partial K((x_0,y_0),r)} v \, ds = v(x_0,y_0)$

i) $y_0 > 0, \ r < |y_0|$:

u harmonična $\Rightarrow$ za u velja LPV

$$\Rightarrow \fint_{\partial K((x_0,y_0),r)} v \, ds \overset{\substack{y_0>0 \\ r<|y_0|}}{=} \int_{\partial K((x_0,y_0),r)} u \, ds = u(x_0,y_0)$$

ii) $y_0 < 0, \ r < |y_0|$:

$$\fint_{\partial K((x_0,y_0),r)} v \, ds \overset{\substack{y_0<0 \\ r<|y_0|}}{=} \fint_{\partial K((x_0,y_0),r)} -u(x,-y) \, ds = \int_0^{2\pi} -u(x_0 + r\cos\varphi, -(y_0 + r\sin\varphi)) \, r \, d\varphi$$

$$\overset{\theta=-\varphi}{=} \int_0^{2\pi} -u(x_0 + r\cos\theta, -y_0 + r\sin\theta) \, r \, d\theta = \fint_{\partial K((x_0,-y_0),r)} -u \, ds$$

$$= -u(x_0, -y_0)$$

iii) $y_0 = 0$:

$$\oint_{\partial K((x_0,y_0),r)} v\, ds = \int_0^{2\pi} v(x_0 + r\cos\varphi,\ r\sin\varphi)$$

$$= \int_0^{\pi} u(x_0 + r\cos\varphi,\ r\sin\varphi)\, r\, d\varphi - \int_{-\pi}^{-2\pi} u(x_0 + r\cos\varphi,\ -r\sin\varphi)\, r\, d\varphi$$

$\theta = -\varphi$

$$= \int_0^{\pi} u(x_0 + r\cos\varphi,\ r\sin\varphi)\, r\, d\varphi + \int_{-\pi}^{-2\pi} u(x_0 + r\cos\theta,\ r\sin\theta)\, r\, d\theta$$

2π-periodičnost

$$= \int_0^{\pi} u(x_0 + r\cos\varphi,\ r\sin\varphi)\, r\, d\varphi - \int_0^{\pi} u(x_0 + r\cos\varphi,\ r\sin\theta)\, r\, d\theta$$

$$= 0 = v(x,0)$$

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2) $B = \mathbb{D} \cap \mathbb{H}$

u zvezna na $\overline{B}$

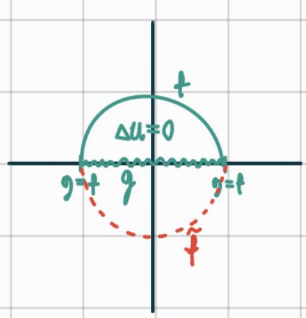
$\Delta u = 0$ na B

$u = f$ na $\partial\mathbb{D} \cap \{y \geq 0\}$

$u = g$ na $\mathbb{D} \cap \{y = 0\}$

$f = g = 0$ v $(\pm 1, 0)$

Izpeljite reprezentacijsko formulo



Kot v prejšnji nalogi želimo razširiti na cel $\mathbb{D}$.

Dobimo nov problem:

$\Delta v_2 = 0$ na $\mathbb{D}$

$v_2 = \tilde{f}$ na $\partial\mathbb{D}$

$$\tilde{f} = \begin{cases} f(x,y) & ; y \geq 0 \\ -f(x,-y) & ; y < 0 \end{cases}$$

$\Delta v_1 = 0$ na $\mathbb{H}$
$v_1 = \tilde{g}$ na $\partial\mathbb{H}$

$$\tilde{g} = \begin{cases} g(x,y) & ; |x| < 1 \\ 0 & ; |x| \geq 1 \end{cases}$$

Ideja 1: Ali $v_1 + v_2$ reši prvotni problem?

$\Delta(v_1 + v_2) = 0$ na B
$v_1 + v_2 = ?$

$\Rightarrow$ Ideja 1 ne deluje ☹

Ideja 2: Rešimo problem za v_1 in z njim popravimo pogoje za v_2.

$v_1 = \frac{y}{\pi} \cdot \int_{-\infty}^{\infty} \frac{1}{(x-s)^2 + y^2} \cdot \tilde{g}(s,0)\, ds$ ... reprezentacijska formula

$\tilde{g} = 0$ za $|x| > 1$

$$\Rightarrow v_1 = \frac{y}{\pi} \int_{-1}^{1} \frac{1}{(x-s)^2 + y^2} \cdot g(s,0)\, ds$$

Oznaka: $\hat{h} = \begin{cases} h(x,y) & ; y \geq 0 \\ -h(x,-y) & ; y < 0 \end{cases}$

Dobimo nov problem za v_2:

$\Delta v_2 = 0$ na $\mathbb{D}$
$v_2 = (f - \hat{v}_1)$ na $\partial\mathbb{D}$

$$\Rightarrow v_2(r \cdot e^{i\theta}) = (P_r * (f - \hat{v}_1))(e^{i\theta})$$

$w = v_2 + v_1$ je rešitev problema

$\Delta w = 0$ na B ✓

$w = f - v_1 + v_1 = f$ na $\partial D \cap \{y \geq 0\}$ ✓

$w = v_1 + v_2 = g + v_2$ na $D \cap \{y = 0\}$?

$\underline{v_2 = 0 \text{ na } D \cap \{y=0\}}$

$$v_2(x,0) \overset{x \geq 0}{=} v_2(re^{i\cdot 0}) = v_2(r) = (P_r * (f \hat{-} v_1))(1)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (f \hat{-} v_1)(\cos t, \sin t)\cdot \underbrace{P_r(-t)}_{\substack{=P_r(t) \\ (\text{cost soda})}}\, dt = *$$

$$P_r(t) = \frac{1-r^2}{1+r^2-2r\cos t}$$

$$* = \frac{1}{2\pi}\left(\int_0^{\pi} \blacksquare + \int_{\pi}^{2\pi} (f \hat{-} v_1)(\cos t, \sin t)\cdot P_r(t)\, dt\right)$$

$$\overset{\substack{-\tau = t \\ -d\tau = dt}}{=} \frac{1}{2\pi}\left(\int_0^{\pi} \blacksquare - \int_{-\pi}^{-2\pi} (f \hat{-} v_1)(\cos\tau, -\sin\tau)\, d\tau\right)$$

$$\overset{\text{definicija } \wedge}{=} \frac{1}{2\pi}\left(\int_0^{\pi} \blacksquare + \int_{-\pi}^{-2\pi} (f - v_1)(\cos\tau, -\sin\tau)\, d\tau\right)$$

$$\overset{2\pi\text{-periodična}}{=} 0$$

$$v(x,0) \overset{x<0}{=} v_2(re^{i\pi})$$

Ampak $P_r(\pi - t) = P_{-r}(t)$ ✓

Torej:

$$u(x,y) = \frac{1}{\pi}\int_{-1}^{1} \frac{y}{(x-t)^2+y^2}\, g(t)\, dt$$

$$+ \frac{1}{2\pi}\int_0^{2\pi}\left[\hat{f}(\cos\tau, \sin\tau) - \frac{1}{\pi}\int_{-1}^{1} \frac{r\sin\tau}{(r\cos\tau - t)^2 + (r\sin\tau)^2}\, g(t)\, dt\right] P_r(\theta - \tau)\, d\tau$$

$$\theta = \operatorname{arctg} \frac{y}{x}$$