1) Dokažite enoličnost rešitve (če obstaja) problema:

$$
\begin{aligned}
&u_{tt} - c^2 u_{xx} = F(t,x) \quad , && 0 < x < L,\ t > 0 \\
&u_x(t,0) = t^2 \quad , && t > 0 \\
&u(t,L) = -t \quad , && t > 0 \\
&u(0,x) = x^2 - L^2 \quad , && 0 \leq x \leq L \\
&u_t(0,x) = \sin^2\left(\tfrac{\pi x}{L}\right) \quad , && 0 \leq x \leq L
\end{aligned}
$$

u_1, u_2 rešitvi

$v := u_2 - u_1$

Dokazujemo: $v \equiv 0$

$$
\begin{aligned}
v_{tt} &= u_{2tt} - u_{1tt} \\
v_{xx} &= u_{2xx} - u_{1xx}
\end{aligned}
$$

$$\Rightarrow v_{tt} - c^2 v_{xx} = u_{2tt} - u_{1tt} - c^2 u_{2xx} + c^2 u_{2xx} = F(t,x) - F(t,x) = 0$$

$$\Rightarrow v_{tt} = c^2 v_{xx}$$

$$
\begin{aligned}
v_x(t,0) &= u_{2x}(t,0) - u_{1x}(t,0) = t^2 - t^2 = 0 \\
v(t,L) &= 0 \\
v(0,x) &= 0 \\
v_t(0,x) &= 0
\end{aligned}
$$

Metoda energij (sicer druga stvar kot pri linearnih PDE ...):

$$\varphi(t) = \int_0^L (v_t^2 + c^2 v_x^2)\, dx \qquad \text{(opazimo / nam pove asistent)}$$

$$\varphi'(t) = \frac{\partial}{\partial t}\int_0^L (v_t^2 + c^2 v_x^2)\,dx$$

φ dovolj lepa

$$= \int_0^L \frac{\partial}{\partial t}(v_t^2 + c^2 v_x^2)\,dx$$

$$= \int_0^L (2 v_t v_{tt} + c^2 \cdot 2 v_x v_{xt})\,dx$$

$$= 2\int_0^L v_t v_{tt}\,dx + 2c^2\int_0^L v_x v_{xt}\,dx$$

per partes

$$= 2\int_0^L v_t v_{tt}\,dx + 2c^2\, v_x v_t \Big|_{x=0}^{x=L} - 2c^2\int_0^L v_{xx} v_t\,dx$$

$v_{tt} = c^2 v_{xx}$

$$= 2c^2\int_0^L v_t v_{xx}\,dx + 2c^2\, v_x v_t \Big|_{x=0}^{x=L} - 2c^2\int_0^L v_{xx} v_t\,dx$$

$$= 2c^2\, v_x v_t \Big|_{x=0}^{x=L}$$

$$= 2c^2\big(v_x(t,L)\underbrace{v_t(t,L)}_{=0} - v_x(t,0)\underbrace{v_t(t,0)}_{=0}\big)$$

$$= 0$$

$\Rightarrow \varphi(t) = \text{konst}$

$v(0,x) = 0 \Rightarrow v_x(0,x) = 0$

$$\varphi(0) = \int_0^L (\underbrace{v_t^2(0,x)}_{=0} + c^2 \underbrace{v_x(0,x)}_{=0})\,dx$$

$$= \int_0^L (0 + c^2 \cdot 0)\,dx$$

$$= 0$$

$\Rightarrow v_t^2 + c^2 v_x^2 = 0$ s.p.

$\Rightarrow v_t, v_x = 0$ s.p.

v dovolj lepa

$\Rightarrow v \equiv \text{konst}$

(dovolj lepa = malo manj kot C^2)

$v(0,x) = 0 \Rightarrow v \equiv 0$

What is a solution to a PDE?

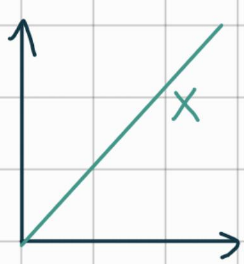
A PDE is an equation of the form $F(x, u, \nabla u, \dots) = 0$.

A solution would be a function u, such that $F(x, u, \nabla u, \dots) = 0$.

Since the solution involves derivatives of u up to some order $m \in \mathbb{N}$, we usually assume that the derivatives exist and are continuous.

Definition: A classical solution on the open set $\Omega \subseteq \mathbb{R}^n$ is $u \in C^m(\Omega)$, such that $F(x, u, \nabla u, \dots) = 0$.

Example: $|\nabla u| = 1$ on $[0,1]$



If we also ask $u(0) = u(1) = 0$, there is no classical solution.

Often we consider Cauchy problems:

$$F(x, u, \nabla u, \dots) = 0 \quad \text{on } \Omega \subseteq \mathbb{R}^n$$
$$g_1(x, u, \nabla u, \dots) = 0 \quad \text{on } \Gamma_1 \subseteq \overline{\Omega}$$
$$\vdots$$
$$g_m(x, u, \nabla u, \dots) = 0 \quad \text{on } \Gamma_m \subseteq \overline{\Omega}$$

In this case, a classical solution is a function $\in C^0(\Omega \cup \Gamma_0 \cup \dots \cup \Gamma_m)$ which satisfies the above equations and is a classical solution to the PDE.

Questions:

- Existence
- Uniqueness

These are related to the regularity and relationship between $F, g_1, \dots, g_m$ on $\Omega, \Gamma_1, \dots, \Gamma_m$.

Example: Quasi-linear equation of 1st order:

F is affine in ∇u

$\Gamma = \Gamma_1$, $g = g_1$ is specified, but usually Ω is not given.

Existence and uniqueness depend on the relationship between F, Γ and g (characteristics).

Example: Heat equation:

F is fixed
$\Omega = (0, \infty) \times I$
$I \subseteq \mathbb{R}$ interval

Usually: $\Gamma = \{0\} \times I \subseteq \partial\Omega$

Example: Wave equation:

F is fixed
$\Omega = \mathbb{R}^2$
$\Gamma_1 = \Gamma_2 = \{t=0\} = \{0\} \times \mathbb{R}$

$g_1(x, u, \nabla u) = u - f(x)$
$g_2(x, u, \nabla u) = \nabla u - g(x)$

For simplicity, take $g \equiv 0$.

$\Rightarrow u = \frac{1}{2}(f(x+t) + f(x-t))$

Nevertheless, this expression makes sense for any f!

So, we still call it a solution to the Cauchy problem.

There are many other types of "solutions". The main ones are weak solutions and viscosity solutions.

Weak solutions:

$u \in C^1(\mathbb{R})$, $\varphi \in C_c^\infty(\mathbb{R})$

$$\int_{-\infty}^{\infty} u'(x)\varphi(x)\,dx = -\int_{-\infty}^{\infty} u(x)\varphi'(x)\,dx \quad \forall \varphi \in C_c^\infty(\mathbb{R})$$

Definition: For any solution u, its weak derivative u' is a function v, such that:

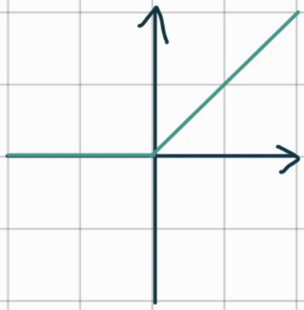
$$\int_{-\infty}^{\infty} v(x)\varphi(x)\,dx = -\int_{-\infty}^{\infty} u(x)\varphi'(x)\,dx$$

Similar for partial derivatives.

Observe: v is defined up to a set of measure zero.

A **weak solution** to $F(x, u, \nabla u) = 0$ is a function u admitting weak derivatives up to order n and such that $F(x, u, \nabla u, \dots) = 0$ almost everywhere.

Example: $u(x) = x \cdot \mathbb{1}_{[0,\infty)}$



$$-\int_{-\infty}^{\infty} u(x)\, \varphi'(x)\, dx = -\int_{0}^{\infty} x\, \varphi'(x)\, dx$$

$$= -\left[x\, \varphi(x)\right]_{x=0}^{x=\infty} + \int_{-\infty}^{\infty} \varphi(x)\, dx = \int_{0}^{\infty} \varphi(x)\, dx$$

$$\Rightarrow u' = v = \mathbb{1}_{(0,\infty)}$$

Definition: $W^{k,p} := \{ u \in L^p ;\ \exists \text{ weak } u^{(j)} \ \&\ u^{(j)} \in L^p \ \forall j \leq l \}$

$$H^k := W^{k,2}$$

Example: $H^1 = W^{1,2} = \{ u \in L^2 ;\ \exists \text{ weak } u' \ \&\ u' \in L^2 \}$

Observe: H^k is a Hilbert space:

$$(u, v)_{H^k} = \sqrt{(u,v)_{L_2}^2 + (u', v')_{L_2}^2 + \dots + (u^{(k)}, v^{(k)})_{L_2}^2}$$

Why is this useful?

H^k is a Hilbert space

Questions:

- Does a weak solution exist?
- Given a weak solution, is it a classical solution?

Theorem (Sobolev):

$$W^{k,p}(\mathbb{R}^n) \subseteq C^r(\mathbb{R}^n) \qquad \forall\, r < k - \frac{n}{p}$$

Example: $p = 2$, $n = 1$

$$u \in H^k \Rightarrow u \in C^r \quad \forall\, r < k - \frac{1}{2}$$

$$(\text{i.e. } r \leq k-1)$$

$$u \in H^1 \Rightarrow u \in C^0$$

Example: $\begin{cases} |u'(x)| = 1 \\ u(0) = u(1) = 0 \end{cases}$

No classical solution

Infinitely many weak solutions

Viscosity solutions:

Cauchy problem:

$$\begin{cases} F(x, u, \nabla u, \dots) = 0 \\ u|_{\partial\Omega} = g \end{cases}$$

Consider:

$$\{ v \in C^n(\Omega) \ ; \ F(x, v, \nabla v, \dots) \geq 0 \ , \ v|_{\partial\Omega} = g(x) \} = A_+$$

$$\{ v \in C^n(\Omega) \ ; \ F(x, v, \nabla v, \dots) \leq 0 \ , \ v|_{\partial\Omega} = g(x) \} = A_-$$

Let:

$$u^+(x) = \inf_{v \in A^+} v(x)$$

$$u^-(x) = \sup_{v \in A^-} v(x)$$

Hope: $u^+ = u^- =: u$

We call such u a viscosity solution.

For this to work, you usually need a maximum principle.

So, if $u_1|_{\partial\Omega} \leq u_2|_{\partial\Omega}$, where $u_1, u_2 \in A^+$, then $u_1 \leq u_2$ on Ω, and similarly for A^-.

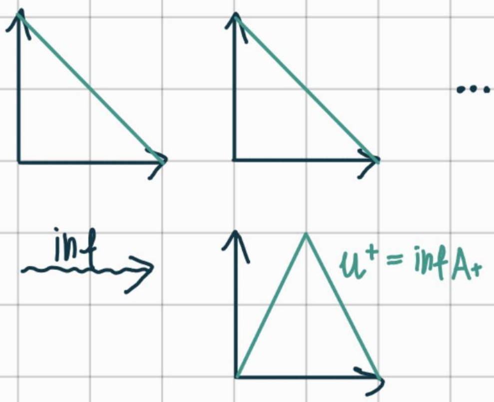
Example: $|u'| = 1$
$u(0) = u(1) = 0$

No classical solution

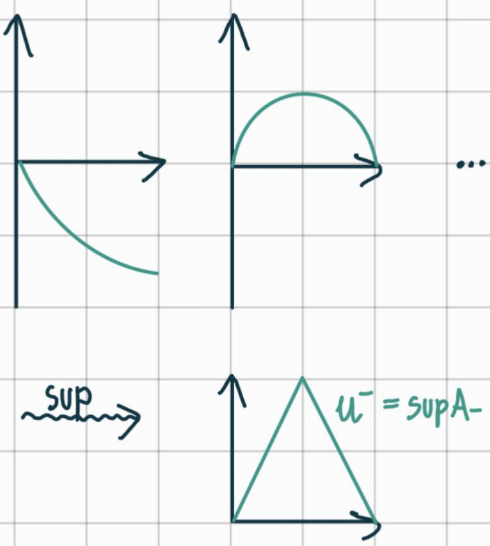
Infinitely many weak solutions

What about viscosity?

$$A^+ = \{ v \in C^1(0,1) \; ; \; |v'| \geq 1, \; v(0) = v(1) \geq 0 \}$$



$$A^- = \{ \dots \}$$



$\Rightarrow u = u^- = u^+$ viscosity solution

7.5.

2a) Dokažite: $$u\left[(y^2 u_x)_x + (x^2 u_y)_y\right] = \nabla \cdot (y^2 u u_x, x^2 u u_y) - \left[(y u_x)^2 + (x u_y)^2\right]$$

$$\frac{\partial}{\partial x}(y^2 u u_x) = y^2 \cdot \frac{\partial}{\partial x}(u u_x) = y^2\left[u_x^2 + u u_{xx}\right]$$

$$\frac{\partial}{\partial y}(x^2 u u_y) = x^2 \cdot \frac{\partial}{\partial y}(u u_y) = x^2\left[u_y^2 + u u_{yy}\right]$$

Vstavimo in pokrajšamo ...

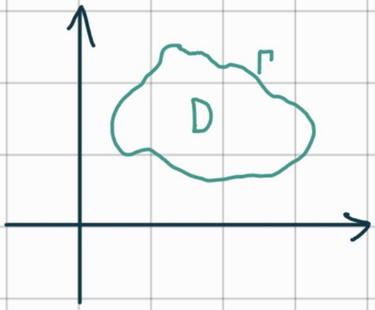
$$\Rightarrow y^2 u u_{xx} + x^2 u u_{yy} = \nabla \cdot (y^2 u u_x, x^2 u u_y) - u\left[(y^2 u_x)_x + (x^2 u_y)_y\right]$$

2b) Naj bo:

$D \subseteq \mathbb{R}^2$
Γ gladek rob

Dokažite enoličnost rešitve problema:

$$(y^2 u_x)_x + (x^2 u_y)_y = F(x,y) \quad , \quad (x,y) \in D$$
$$u(x,y) = f(x,y) \quad ; \quad (x,y) \in \Gamma$$



u_1, u_2 rešitvi

$v := u_2 - u_1$

$$\Rightarrow (y^2 v_x)_x + (x^2 v_y)_y = 0 \quad \text{na } D$$
$$v(x,y) = 0 \quad \text{na } \Gamma$$

$$\varphi = \iint_D (y^2 v_x^2 + x^2 v_y^2)\, dx\, dy$$
$$= \iint_D \Big[\nabla\cdot(y^2 v v_x, x^2 v v_y) - v\underbrace{\big[(y^2 v_x)_x + (x^2 v_y)_y\big]}_{=0 \text{ na } D}\Big]\, dx\, dy = *$$

Green: $\displaystyle\iint_D \nabla\cdot(F_1,F_2)\, dx\, dy = \int_\Gamma (F_1,F_2)\cdot\vec{n}\, ds$

$$\iint_D \underset{\nabla\cdot(-Y,X)}{\underset{\parallel}{(X_y - Y_x)}}\, dx\, dy = \int_\Gamma \underset{(-Y,X)}{\underset{\parallel}{(X,Y)}}\, dy$$

$$* = \iint_D \nabla\cdot(y^2 v v_x, x^2 v v_y)\, dx\, dy$$

$$= \int_\Gamma (y^2 \underset{0}{\underset{\parallel}{v}} v_x, x^2 \underset{0}{\underset{\parallel}{v}} v_y) \cdot \vec{n}\, ds$$

robni pogoji

$$= 0$$

$$\Rightarrow \varphi = 0 \text{ na } D$$

$$\Rightarrow y^2 v_x^2 + x^2 v_y^2 = 0 \text{ na } D$$

$$\Rightarrow \nabla v = 0 \text{ na } D$$

$$\Rightarrow v = \text{konst}$$

$$v(x,y) = 0 \Rightarrow v = 0$$

Princip maksima za toplotno enačbo:

Naj bo $\Omega = (0,\infty) \times (0,b)$ in $v : \Omega \to \mathbb{R}$, $v = v(t,x)$ zvezno odvedljiva dvakrat po x ter enkrat po t.

Za $T > 0$ naj bo $\Omega_T = (0,T) \times (0,b)$ in $\partial' \Omega_T := \partial \Omega_T \setminus (\{T\} \times (0,b))$ parabolični rob.

Če na Ω velja $v_t - v_{xx} \leq 0$, tedaj velja:

$$\max_{\overline{\Omega_T}} v = \max_{\partial' \Omega_T} v$$

Dokaz: i) Predpostavimo: $v_t - v_{xx} < 0$ na Ω_T

$$\Rightarrow \exists (t_0, x_0) \in \overline{\Omega_T} : \max_{\overline{\Omega_T}} v(t,x) = v(t_0, x_0)$$

- $(t_0, x_0) \in \Omega_T$

$\Rightarrow v_t(t_0,x_0) = 0$

$\Rightarrow v_x(t_0,x_0) = -v_{xx}(t_0,x_0) < 0$

$\Rightarrow v_{xx}(t_0,x_0) > 0$

To ni mogoče, saj to pomeni, da je x_0 minimum od $x \mapsto v(t_0,x)$

- $t_0 = T \wedge x_0 \in (0,b)$

$\Rightarrow x \mapsto v(T,x)$ ima minimum v $x = x_0$

$\Rightarrow u_{xx}(T,x_0) \leq 0$

$\Rightarrow v_t(T,x_0) < v_{xx}(T,x_0) \leq 0$

$\Rightarrow t \mapsto v(t,x_0)$ se zmanjšuje za $t \in (T-\varepsilon, T]$

To ni mogoče, ker potem $v(T,x_0)$ ni maksimum

$\Rightarrow (t_0,x_0) \in \partial' \Omega_T$

ii) Predpostavimo: $v_t - v_{xx} = 0$

Obravnavajmo $v_\varepsilon(t,x) = v(t,x) - \varepsilon t$, $\varepsilon > 0$

$\Rightarrow (v_\varepsilon)_t - (v_\varepsilon)_{xx} = v_t - v_{xx} - \varepsilon < 0$

$$\Rightarrow \max_{\overline{\Omega}_T} v = \max_{\overline{\Omega}_T} (v_\varepsilon + \varepsilon t) \leq \max_{\partial'\Omega_T} v_\varepsilon + \varepsilon T$$

$$\overset{v_\varepsilon \leq v}{\leq} \max_{\partial'\Omega_T} v + \varepsilon T \quad \forall \varepsilon > 0$$

$$\Rightarrow \max_{\overline{\Omega}_T} v \leq \max_{\partial'\Omega_T} v$$

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3) Naj bo:

$u, v \in C^2((0,T)\times(0,L)) \cap C([0,T]\times[0,L])$

$u_t - u_{xx} \leq v_t - v_{xx}$ na $(0,T)\times(0,L)$

$u \leq v$ na paraboličnem robu Γ_p

$\Gamma_p = \{0\}\times[0,L] \cup [0,T]\times\{0,L\}$

Dokažite: $u \leq v$ na $[0,T]\times[0,L] = \Omega_T$

$u_t - v_t - u_{xx} + v_{xx} = (u-v)_t - (u-v)_{xx} \leq 0$

Na paraboličnem robu: $u - v \leq 0$

$$\overset{\text{izrek}}{\Rightarrow} \max_{\overline{\Omega}_{T-\varepsilon}} (u-v) = \max_{\partial'\Omega_{T-\varepsilon}} (u-v) \quad \forall \varepsilon > 0$$

Nimamo (še) odvedljivosti za $t = T$, zato smo morali območje malo zmanjšati.

Ampak $u - v$ je zvezna.

$$\overset{\text{zveznost}}{\Rightarrow} \max_{\overline{\Omega}_T} (u-v) = \max_{\partial'\Omega_T} (u-v) \leq 0$$

$\Rightarrow u - v \leq 0$ na $\overline{\Omega}_T$

$\Rightarrow u \leq v$ na $\overline{\Omega}_T$

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4) Dokažite: Harmonične funkcije dveh spremenljivk nimajo izoliranih ničel

$u_{xx} + u_{yy} = 0$

$u(x,y) = \fint_{S_r} u(\alpha,\beta)\, dS_{\alpha,\beta} = \frac{1}{2\pi r}\int_{S_r} u(\alpha,\beta)\, dS_{\alpha,\beta}$

$u(x_0,y_0) = 0$

$u(x_0,y_0) = \frac{1}{2\pi r}\int_{S_r} u(\alpha,\beta)\, dS_{\alpha,\beta} = \frac{1}{2\pi r}\int_0^1 u(\gamma(t)) \cdot |\dot{\gamma}(t)|\, dt$

Denimo: (x_0, y_0) izolirana

$\Rightarrow \exists r > 0 : \; u(\gamma(t)) \neq 0 \quad \forall t$

zvezna

$\Rightarrow \frac{1}{2\pi r}\int_0^1 u(\gamma(t)) \cdot |\underset{\substack{\vee \\ 0}}{\dot{\gamma}(t)}|\, dt \neq 0$

$\times$