1) $u_{tt} - u_{xx} = 0$

$t \in (0, \infty)$

$x \in (0, \pi)$

$u(0,x) = f(x)$

$u_t(0,x) = 0$

$u_x(t,0) + \alpha u(t,0) = u_x(t,\pi) + \alpha u(t,\pi) = 0$

$\alpha \neq 0$

$f \in C^1$

$u(t,x) = T(t) \cdot X(x)$

$u_{tt}(t,x) = T''(t) \cdot X(x)$

$u_{xx}(t,x) = T(t) \cdot X''(x)$

$\Rightarrow T''(t) \cdot X(x) - T(t) \cdot X''(x) = 0 \quad / \cdot \frac{1}{TX}$

$\Rightarrow \frac{T''(t)}{T(t)} - \frac{X''(x)}{X(x)} = 0$

$\Rightarrow \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$

$V = \{ X \in L^2(0,\pi) \ ; \ X'(0) + \alpha X(0) = 0, \ X'(\pi) + \alpha X(\pi) = 0 \}$

$X(x) = Ae^{\sqrt{\lambda} x} + Be^{-\sqrt{\lambda} x}$

$X'(x) = A\sqrt{\lambda} e^{\sqrt{\lambda} x} - B\sqrt{\lambda} e^{-\sqrt{\lambda} x}$

$X'(0) + \alpha X(0) = A\sqrt{\lambda} - B\sqrt{\lambda} + \alpha(A+B) = 0 = \sqrt{\lambda}(A-B) + \alpha(A+B)$

$$X'(\pi)+X(\pi) = A\sqrt{\lambda}e^{\sqrt{\lambda}\pi} - B\sqrt{\lambda}e^{-\sqrt{\lambda}\pi} + \alpha(Ae^{\sqrt{\lambda}\pi} + Be^{-\sqrt{\lambda}\pi}) = 0$$

$$A(\sqrt{\lambda}+\alpha) = B(\sqrt{\lambda}-\alpha)$$

$$A(\sqrt{\lambda}e^{\sqrt{\lambda}\pi} + \alpha e^{\sqrt{\lambda}\pi}) = B(\sqrt{\lambda}e^{-\sqrt{\lambda}\pi} - \alpha e^{-\sqrt{\lambda}}) \quad / \cdot e^{\sqrt{\lambda}\pi}$$

$$Ae^{2\pi\sqrt{\lambda}}(\sqrt{\lambda}+\alpha) = B(\sqrt{\lambda}-\alpha) = A(\sqrt{\lambda}+\alpha)$$

i) $A=0$:

$$\sqrt{\lambda}-\alpha = 0$$

$$\Rightarrow \sqrt{\lambda} = \alpha$$

$$\Rightarrow \lambda = \alpha^2$$

ii) $\sqrt{\lambda}-\alpha = 0$:

- $B=0$:

$$\lambda = \alpha^2$$

- $\sqrt{\lambda} = \alpha$:

Ni mogoče

iii) $e^{2\pi\sqrt{\lambda}} = 1$:

$$2\pi\sqrt{\lambda} = 2\pi i \cdot k \ , \quad k \in \mathbb{Z}$$

$$\Rightarrow \lambda = -k^2$$

Torej:

i) $X_{-\alpha}(x) = e^{-\alpha x}$

ii) $X_{\alpha}(x) = e^{\alpha x}$

iii) $\lambda = -k^2$

$$A = ik - \alpha$$
$$B = ik + \alpha$$

$$\tilde{X}_k(x) = (ik-\alpha)e^{ikx} + (ik+\alpha)e^{-ikx}$$

$$= (ik-\alpha)(\cos(kx) + i\sin(kx)) + (ik+\alpha)(\cos(-kx) + i\sin(-kx))$$

$$= 2ik\cos(kx) - 2i\alpha\sin(kx)) \qquad / \cdot \frac{i}{2}$$

$$X_k(x) = -k\cos(kx) + \alpha\sin(kx)$$

Drug način:

$$X_k(x) = \tilde{A}\cos(kx) + \tilde{B}\sin(kx)$$

$$\begin{cases} X_k'(0) + \alpha X_k(0) = 0 \\ X_k'(\pi) + \alpha X_k(\pi) = 0 \end{cases}$$

$$u(t,x) = \sum_{k=1}^{\infty} T_k(t)X_k(x) + T_\alpha(t)X_\alpha(x) + T_{-\alpha}(t)X_{-\alpha}(x)$$

$$T_k'' = -k^2 T_k$$
$$T_{\pm\alpha}'' = \alpha^2 T_{\pm\alpha}$$

$$T_k = C_k \cos(kt) + D_k \sin(kt)$$
$$T_\alpha = C_\alpha e^{\alpha t} + D_\alpha e^{-\alpha t}$$
$$T_{-\alpha} = C_{-\alpha} e^{\alpha t} + D_{-\alpha} e^{-\alpha t}$$

$$u_t(t,x) = \sum_{k=1}^{\infty} T_k'(t) X_k(x) + T_\alpha'(t) X_\alpha(x) + T_{-\alpha}'(t) X_{-\alpha}(x)$$

$$u(0,x) = f(x)$$
$$= \sum_{k=0}^{\infty} T_k(0) X_k(x) + T_\alpha(0) X_\alpha(x) + T_{-\alpha}(0) X_{-\alpha}(x)$$
$$= \sum_{k=1}^{\infty} C_k X_k(x) + (C_\alpha + D_\alpha) X_\alpha(x) + (C_{-\alpha} + D_{-\alpha}) X_{-\alpha}(x)$$

$$u_t(0,x) = 0$$
$$= \sum_{k=1}^{\infty} k D_k X_k(x) + (C_\alpha \alpha - D_\alpha \alpha) X_\alpha(x) + (C_{-\alpha}\alpha - D_{-\alpha}\alpha) X_{-\alpha}(x)$$

$$\langle f, X_j \rangle = C_j \cdot \|X_j\|^2$$

$$\langle 0, X_j \rangle = 0 = k \cdot D_j \cdot \|X_j\|^2 \Rightarrow D_j = 0$$

Opomba: A simetrična

$Av = \lambda v$

$Aw = \mu w$

$\lambda \neq \mu$

$$\lambda \langle v, w \rangle = \langle \lambda v, w \rangle = \langle Av, w \rangle$$
$$= \langle v, A^T w \rangle \overset{\text{sim.}}{=} \langle v, Aw \rangle$$
$$= \langle v, \mu w \rangle = \mu \langle v, w \rangle$$
$$\Rightarrow \langle v, w \rangle = 0 \qquad (\text{ali } \lambda = \mu)$$

$\Rightarrow v \perp w$

$$\langle f, X_\alpha\rangle = (C_\alpha + D_\alpha)\|X_\alpha\|^2 + (C_{-\alpha} + D_{-\alpha})\langle X_\alpha, X_{-\alpha}\rangle$$

$$\langle 0, X_\alpha\rangle = (C_\alpha \cdot \alpha + D_\alpha \cdot \alpha)\|X_\alpha\|^2 + (C_{-\alpha} \cdot \alpha + D_{-\alpha} \cdot \alpha)\langle X_\alpha, X_{-\alpha}\rangle$$

$$\langle f, X_{-\alpha}\rangle = (C_\alpha + D_\alpha)\langle X_\alpha, X_{-\alpha}\rangle + (C_{-\alpha} + D_{-\alpha})\|X_{-\alpha}\|^2$$

$$\langle 0, X_{-\alpha}\rangle = (C_\alpha \cdot \alpha - D_\alpha \cdot \alpha)\langle X_\alpha, X_{-\alpha}\rangle + (C_{-\alpha} \cdot \alpha - D_{-\alpha} \cdot \alpha)\|X_{-\alpha}\|^2$$

Poračunamo koeficiente iz sistema teh enačb ...

24.4.

2) $u_t + 2u = u_{xx} + \sin x$
$t \in (0, \infty)$
$x \in [0, \pi]$
$u(0,x) = \sin x$
$u(t,0) = u(t,\pi) = 0$

$$u(t,x) = T(t)X(x)$$

$$u_t = T'(t)X(x)$$
$$u_{xx} = T(t)X''(x)$$

$$T'(t)X(x) + 2T(t)X(x) = T(t)X''(x) + \underline{\underline{\sin x}}$$:(

$$u = v + \alpha \sin x$$

$$v_t + 0 + 2v + 2\alpha \sin x = v_{xx} - \alpha \sin x + \sin x$$

$$v_t + 2v - v_{xx} = -3\alpha \sin x + \sin x$$

$\alpha := \frac{1}{3}$

$\Rightarrow v_t + 2v - v_{xx} = 0$ ... homogena

$u = v + \frac{1}{3}\sin x$

$u(0,x) = \sin x$
$\parallel$
$v(0,x) + \frac{1}{3}\sin x$

$\Rightarrow v(0,x) = \frac{2}{3}\sin x$

$u(t,0) = u(t,\pi) = 0$
$\parallel \quad \parallel$
$v(t,0) \quad v(t,\pi)$

$v(t,x) = T(t)X(x)$

$T'(t)X(x) + 2T(t)X(x) - T(t)X''(x) = 0$

$\frac{T'(t)}{T(t)} + 2 - \frac{X''(x)}{X(x)} = 0$

$\frac{T'(t)}{T(t)} + 2 = \frac{X''(x)}{X(x)} =: \lambda$

$\frac{T'(t)}{T(t)} = \lambda - 2 \Rightarrow T(t) = e^{(\lambda-2)t}$

$X''(x) = \lambda X(x) \Rightarrow X_k(x) = \sin(kx)$

$(Ax = y \quad , \quad x \in \{A^{-1}y + \ker A\}$

$v(t,x) = \sum_{k=1}^{\infty} c_k e^{(-k^2-2)t} \sin(kx)$

$$v(0,x) = \sum_{k=1}^{\infty} c_k \sin(kx) = \frac{2}{3}\sin x$$

$$\Rightarrow c_1 = \frac{2}{3}$$

$$c_k = 0 \quad \text{za } i > 1$$

$$\Rightarrow v(t,x) = \frac{2}{3}\sin x \, e^{(-1^2-2)t}$$

$$\Rightarrow u(t,x) = \frac{2}{3}\sin x \, e^{-3t} + \frac{1}{3}\sin x$$

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3) $u_t = u_{xx}$
$t \in (0,\infty)$
$x \in [0,1]$
$u(0,x) = 1-x$
$u_t(t,0) = u(t,1) = 0$

$u = T(t)X(x)$
$u_t = T'(t)X(x)$
$u_{xx} = T(t)X''(x)$

$$T'(t)X(x) = T(t)X''(x)$$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

$\lambda = 0$:

$T'(t) = 0 \Rightarrow T(t) = C$
$X''(x) = 0 \Rightarrow X(x) = Ax + B$

$X'(x) = A = 0$

$$X(1) = A + B = 0 \Rightarrow B = 0$$

$\Rightarrow \lambda = 0$ ni lastna vrednost

$$T(t) = C \cdot e^{\lambda t}$$

$$X(x) = A\sin(\sqrt{-\lambda}\,x) + B\cos(\sqrt{-\lambda}\,x)$$

$$X'(x) = 0 = A\sqrt{-\lambda} \Rightarrow A = 0$$

$$X(1) = 0 = B\cos(\sqrt{-\lambda}) \Rightarrow \sqrt{-\lambda} = k\pi + \frac{\pi}{2}$$

$$\Rightarrow \lambda_k = -\left(k\pi + \frac{\pi}{2}\right)^2$$

$$\Rightarrow X_k(x) = \cos\left(\left(k\pi + \frac{\pi}{2}\right)x\right)$$

$$\Rightarrow u(t,x) = \sum_{k=0}^{\infty} C_k\, e^{-(k\pi + \frac{\pi}{2})^2 t} \cos\left(\left(k\pi + \frac{\pi}{2}\right)x\right)$$

$$\langle 1-x, X_j \rangle = C_k \cdot \|X_j\|^2$$

$$\Rightarrow C_k = \frac{\int_0^1 (1-x)\cos\left(\left(\frac{\pi}{2} + k\pi\right)x\right)dx}{\int_0^1 \cos\left(\left(\frac{\pi}{2} + k\pi\right)x\right)^2 dx} = \ldots = \frac{8}{(2\pi k + \pi)^2}$$