27.10.

1) $K = \{$zgornjetrikotne 2×2 matrike nad $F\}$

$$M = \begin{bmatrix} F \\ F \end{bmatrix}$$

Poišči vse podmodule M.

M K-modul, $K' \subseteq K \Rightarrow M$ K'-modul

F podkolobar $K \Rightarrow F$-podmoduli so vek. podpr. nad F

${}_K M$ = "levi K-modul M"

${}_K N \leq M$

Kandidati:

- $\{0\}, F^2 = M$ ✓
- $N = \{\lambda \begin{bmatrix} u \\ v \end{bmatrix} ; \lambda \in F\}$, $u, v \in F$

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \cdot \lambda \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \underset{\in N}{\begin{bmatrix} au + bv \\ vc \end{bmatrix}} \overset{?}{=} \begin{bmatrix} u \\ v \end{bmatrix}$$

$\lambda au + \lambda bv = \mu u$
$\lambda vc = \mu$

$\Rightarrow \lambda c = \mu$

$\lambda au + \lambda bv = \lambda cu$
$au + bv = cu$
$\forall a, b, c \in F$
$\times$

$v = 0 : \lambda a u = \mu u$

$\Rightarrow$ K-podmodul samo če $\begin{bmatrix} F \\ 0 \end{bmatrix}$

---

2) $M = {}_{\mathbb{Z}}\mathbb{Z}_{12} = \{0, 1, \ldots, 11\}$

Poišči vse $\mathbb{Z}$-podmodule M. Kateri so enostavni?

Modul $N \neq 0$ je enostaven, če sta 0 in N edina njegova podmodula.

$\mathbb{Z}_2 = \{0, 6\}$ je enostaven

$\mathbb{Z}_3 = \{0, 4, 8\}$ je enostaven

$\mathbb{Z}_4 = \{0, 3, 6, 9\}$ ni enostaven, ker vsebuje $\mathbb{Z}_2$

$\mathbb{Z}_6 = \{0, 2, 4, 6, 8, 10\}$ ni enostaven, ker vsebuje $\mathbb{Z}_2$

---

3) Določi $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_{12})$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_3, \mathbb{Z}_4)$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{12}, \mathbb{Z}_{12})$.

a) $\mathrm{Hom}({}_{\mathbb{Z}}\mathbb{Z}, {}_{\mathbb{Z}}\mathbb{Z}_{12})$

$\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}_{12}$
$1 \mapsto a$

$\varphi(n) = \varphi(n \cdot 1) = n \cdot \varphi(1)$

Recimo $a = 11$: $\varphi(1) = 11$

$\Rightarrow \varphi(n) = n \cdot 11 \mod 12$

$\varphi(2) = 22 \equiv 10 \mod 12$
$\varphi(3) = 33 \equiv 9 \mod 12$
$\varphi(4) = 44 \equiv 8 \mod 12$

$\Rightarrow \varphi(\mathbb{Z}) = \mathbb{Z}_{12} \leq \mathbb{Z}_{12}$, saj $(11) = \mathbb{Z}_{12}$

$$\boxed{\begin{array}{l} K \text{ kolobar} \\ {}_K M \text{ modul} \\ \\ \Rightarrow \operatorname{Hom}(K,M) = \left\{ \begin{array}{l} \varphi : K \to M \\ \quad x \mapsto x \cdot m \end{array} ; \; m \in M \right\} \cong {}_K M \end{array}}$$

Recimo $a = 0$: $\varphi(1) = 0$

$\Rightarrow \varphi(\mathbb{Z}) = 0$

$\operatorname{Hom}(\mathbb{Z}, \mathbb{Z}_{12})$ je $\mathbb{Z}$-modul

$x \in \mathbb{Z}$, $\varphi \in \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}_{12})$

$x \cdot \varphi_a = (n \mapsto x \cdot \varphi_a(n)) = (n \mapsto \varphi_{ax}(n))$

$\operatorname{Hom}(\mathbb{Z}, \mathbb{Z}_{12}) \cong \mathbb{Z}_{12}$

$$\begin{aligned} \psi : \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}_{12}) &\to \mathbb{Z}_{12} \\ \varphi_a &\mapsto a \end{aligned}$$

$\psi(\varphi_a + \varphi_b) = a + b$
$\psi(x\varphi_a) = xa$

b) $\operatorname{Hom}({}_{\mathbb{Z}}\mathbb{Z}_3, {}_{\mathbb{Z}}\mathbb{Z}_4) = H$

$\varphi \in H$

$\varphi: \mathbb{Z}_3 \to \mathbb{Z}_4$

$\varphi(1) = a$

$\varphi(n) = n \cdot a$

$\overset{\mathbb{Z} \ni}{3} \cdot 1 = 0$

$\varphi(3 \cdot 1) = 3 \cdot a \mod 4$

$\parallel$

$\varphi(0) = 0 \cdot a = 0 \mod 4$

$\Rightarrow a = 0 \Rightarrow \varphi(x) \equiv 0 \Rightarrow H = \{0\}$

c) $\mathrm{Hom}({}_{\mathbb{Z}}\mathbb{Z}_{12}, {}_{\mathbb{Z}}\mathbb{Z}_{12}) = H$

$\mathbb{Z} \xrightarrow{\varphi_a \text{ hom.}} \mathbb{Z}_{12}$

${}_{\mathbb{Z}}\ker \varphi_a = \{n \in \mathbb{Z} \,;\, \varphi_a(n) = 0\} = \{\overset{\mathbb{Z} \ni}{n} \,;\, na = 0 \mod 12\} \geq_{\mathbb{Z}} 12\mathbb{Z}$

φ_a porodi $\overline{\varphi_a}$:

$\overset{\mathbb{Z}_{12} \,\parallel}{{}_{\mathbb{Z}}\mathbb{Z} / {}_{\mathbb{Z}}12\mathbb{Z}} \xrightarrow{\overline{\varphi_a}} \mathbb{Z}_{12}$

$n + 12\mathbb{Z} \mapsto \varphi_a(n)$

$\Rightarrow H = \mathbb{Z}_{12}$

$\mathrm{Hom}(\mathbb{Z}_{12}, \mathbb{Z}_{12}) = \mathrm{End}(\mathbb{Z}_{12})$

$$\boxed{\begin{array}{c} C \leq A,\ \varphi(C) = 0 \\ \begin{tikzcd} A \arrow[d, "pr"'] \arrow[dr, bend left, "\varphi"] & \\ \text{„}A/C\text{"} \arrow[r, "\overline{\varphi}"] & B \end{tikzcd} \end{array}}$$

---

4) Dokaži: ${}_K K$ enostaven $\Leftrightarrow$ K obseg

($\Rightarrow$) $K, \{0\}$ edina leva K-modula

$a \in K,\ a \neq 0$

a obrnljiv

$Ka = \{xa\ ;\ x \in K\}$

i) $Ka = \{0\}$:

$1 \cdot a = a \neq 0$

$\times$

ii) $Ka = K$:

$1 = xa$

Zakaj je x desni inverz?

$K \xrightarrow{\varphi_a} K$
$x \mapsto xa$

$\varphi_a \in End({}_K K)$

3.11.

1) Dokaži: Neničelni modul M je enostaven $\Leftrightarrow$ $M = Km$ $\forall m \in M$

($\Rightarrow$) Ako bi $Km \subsetneq M$ bi imeli netrivialni pravi podmodul, razen za $m=0$.

($\Leftarrow$) Denimo $N \leq M$, kjer je $N \neq 0$ in $N \neq M$.

N vsebuje $u \in M \setminus \{0\}$. Zaradi zaprtosti vsebuje vse elemente M, saj je $Ku = M$.

$\Rightarrow M = Ku \leq N \subsetneq M$

$\cancel{\quad}$

---

2) Dokaži: Modul ${}_KM$ je enostaven, če in samo če $M \cong K/{}_KI$, kjer je ${}_KI \leq K$ maksimalen levi ideal.

($\Rightarrow$) Naj bo M enostaven.

$$\varphi : K \to M$$
$$x \mapsto xm \quad , \quad m \in M \setminus \{0\}$$

Po prejšnji nalogi φ je surjektiven.

$\Rightarrow M \cong K/\ker\varphi = K/I$

$xm + ym = (x+y)m$
$y(xm) = (yx)m$

$\Rightarrow I \leq K$

<u>I maksimalen</u>

Denimo, da je $I \subsetneq J \subsetneq K$.

$$\begin{tikzcd}
K \arrow[d, dash] & M \arrow[d, dash] \\
J \arrow[r, leftrightarrow, "\text{bij.}"] \arrow[d, dash] & \varphi(J) \arrow[d, dash] \\
\ker\varphi & 0
\end{tikzcd}$$

$0 \subsetneq \varphi(J) \subsetneq M$

$I \subsetneq J \Rightarrow 0 \subsetneq \varphi(J)$

Denimo $\varphi(J) = M$.

$m \in M: \; m = \varphi(J) = \{xm \,;\, x \in J\}$

V posebnem: $x = x \cdot m$, $x \in J$, $I \subseteq J$

$m - xm = 0$

$(1-x)m = 0$

$1 - x \in \ker\varphi = I \subseteq J$

$\Rightarrow 1 = (1-x) + x \in J$

$\Rightarrow J = K$

$\times$

$\Rightarrow \varphi(J) \neq M$

M enostaven

$\times$

$(\Leftarrow)$ $0 \subsetneq N \subseteq M$

$$\begin{array}{ccccc} 0 & \subsetneq & N & \subseteq & M \\ \varphi\uparrow & & \varphi\uparrow & & \varphi\uparrow \\ \ker\varphi & \subsetneq & J & \subseteq & K \end{array}$$

$J = \varphi^*(N)$

$\varphi(J) = N$

$(\subseteq)$ Očitno

$(\supseteq)$ $n \in N$

$n = xm, \quad x \in K$

$J = \varphi^*(N)$

$\Rightarrow x \in J$

$\Rightarrow n \in \varphi(J)$

Če $M \cong K/I = \{x \cdot (1+I) \; ; \; x \in K\}$:

$K/I \to M$

$1+I \mapsto m \in M \setminus \{0\}$

$x+I = x(1+I) \mapsto xm$

$\varphi : K \to M$

$x \mapsto xm$

$$\begin{tikzcd}
K \arrow[d, dash] & & M \arrow[d, dash] \\
J_1 \arrow[d, dash] & \longrightarrow & \varphi(J_1) \arrow[d, equal] \\
J_2 \arrow[d, dash] & & \varphi(J_2) \arrow[d, dash] \\
\ker\varphi & & 0
\end{tikzcd}$$

$$\begin{tikzcd}
K \arrow[d, dash] & & M/\varphi(J_2) \arrow[d, dash] \\
J_1 \arrow[d, dash] & \longrightarrow & \varphi(J_1)/\varphi(J_2) = 0 \arrow[d, dash] \\
J_2 & & 0
\end{tikzcd}$$

Enostavni $\mathbb{Z}$-moduli:

max ideali: $(2), (3), (5), \ldots$

$\downarrow \psi$

enostavni moduli: $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5, \ldots$

Naloga 2 $\Rightarrow$ ψ je surjektivna

K komutativen $\Rightarrow$ ψ je bijektivna

${}_K I, {}_K J \overset{max}{\trianglelefteq} K$, $K/I \cong K/J \overset{?}{\Rightarrow} I = J$ *

$\operatorname{ann}(M) := \{x \in K \,;\, x \cdot m = 0 \ \forall m \in M\} = \bigcap_{m \in M} \operatorname{ann}(m) \triangleleft K$

$M \xrightarrow[\varphi]{\cong} N \Rightarrow \operatorname{ann}(M) = \operatorname{ann}(N)$

$\operatorname{ann}(N) = \{x \in K \,;\, x \cdot n = 0 \ \forall \underset{n = \varphi(m)}{n \in N}\}$

$\overset{\varphi \text{ surj}}{=} \{x \in K \,;\, x\varphi(m) = 0 \ \forall m \in M\}$

$$= \{x \in K ;\ \varphi(xm) = 0\ \ \forall m \in M\}$$

$$\overset{\varphi \text{ inj}}{=} \{x \in K ;\ xm = 0\ \ \forall m \in M\} = \operatorname{ann}(M)$$

$$\varphi(x \cdot m) = x\varphi(m)$$

$$\operatorname{ann}(K/I) = \{x ;\ x \cdot (y+I) = 0\ \ \forall y \in K\}$$

$$= \{x ;\ xy + I = 0\ \ \forall y \in K\}$$

$$= \{x ;\ xy \in I\ \ \forall y \in K\} \overset{y=1}{\subseteq} I$$

K komutativen $\Rightarrow$ $x \in_K I : xy = y\underset{\in}{\overset{{}_K I}{x}} \in I \Rightarrow I \subseteq \operatorname{ann}(K/I)$

(dovolj: $I \triangleleft K$)

$\Rightarrow$ * velja: $I = J$

$K = M_2(F)$

levi ideali:

$$\begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}, \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}, \dots$$

enostavni moduli:

$$\begin{bmatrix} F \\ F \end{bmatrix}$$

---

3) Katerim kolobarjem so izomorfni $\operatorname{End}_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z})$, $\operatorname{End}_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_3)$, $\operatorname{End}_{\mathbb{Z}}(\mathbb{Z}_6 \oplus \mathbb{Z}_3)$?

a) $\operatorname{End}_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z})$

Generatorja: $(0,1), (1,0)$

$$\varphi : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$
$$(0,1) \longmapsto (a,b)$$
$$(1,0) \longmapsto (c,d)$$

$a,b,c,d \in \mathbb{Z}$

$$\psi : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$
$$(0,1) \longmapsto (x,y)$$
$$(1,0) \longmapsto (z,w)$$

$x,y,z,w \in \mathbb{Z}$

$$(\psi \circ \varphi)(0,1) = \psi(a,b) = (bx + az, by + aw) = *$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} bx + az \\ by + aw \end{bmatrix} = \begin{bmatrix} z & x \\ w & y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$* = \begin{bmatrix} z & x \\ w & y \end{bmatrix} \begin{bmatrix} c & a \\ d & b \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \mathrm{End}_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}) \cong M_2(\mathbb{Z})$$

$$\varphi \longmapsto \begin{bmatrix} pr_1(\varphi(1,0)) & pr_1(\varphi(0,1)) \\ pr_2(\varphi(1,0)) & pr_2(\varphi(0,1)) \end{bmatrix} = \begin{bmatrix} pr_1(\varphi(\iota_1(1))) & pr_1(\varphi(\iota_2(1))) \\ pr_2(\varphi(\iota_1(1))) & pr_2(\varphi(\iota_2(1))) \end{bmatrix}$$

- $\mathrm{End}_{\mathbb{Z}}(K \oplus K) \cong M_2(K)$

$pr_1 : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$
$(a, b) \mapsto a$

$\iota_1 : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$
$a \mapsto (a, 0)$

$\Rightarrow (pr_i \circ \varphi \circ \iota_j)(1)$

$\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Z}$

$\psi \mapsto \psi(1)$
$(x \mapsto xa) \leftarrow a$

- $\underline{\mathrm{End}_K({}_K M \oplus {}_K M) = M_2(\mathrm{End}(M))}$

- $\underline{\mathrm{End}_K({}_K M \oplus {}_K N)} = \begin{bmatrix} \mathrm{End}(M) & \mathrm{Hom}(N,M) \\ \mathrm{Hom}(M,N) & \mathrm{End}(N) \end{bmatrix}$

b) $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_3) = \begin{bmatrix} \mathrm{End}(\mathbb{Z}_2) & \mathrm{Hom}(\mathbb{Z}_3, \mathbb{Z}_2) \\ \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_3) & \mathrm{End}(\mathbb{Z}_3) \end{bmatrix} = \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & \mathbb{Z}_3 \end{bmatrix} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$

c) $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_6 \oplus \mathbb{Z}_3) = \begin{bmatrix} \overset{\mathbb{Z}_6}{\overset{\shortparallel}{\mathrm{End}(\mathbb{Z}_6)}} & \mathrm{Hom}(\mathbb{Z}_3, \mathbb{Z}_6) \\ \mathrm{Hom}(\mathbb{Z}_6, \mathbb{Z}_3) & \underset{\mathbb{Z}_3}{\underset{\shortparallel}{\mathrm{End}(\mathbb{Z}_3)}} \end{bmatrix} = *$

$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \to \mathbb{Z}_3$

${}_{\mathbb{Z}}\mathrm{Hom}(\mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z}_3) \cong \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_3) \oplus \mathrm{Hom}(\mathbb{Z}_3, \mathbb{Z}_3) \cong \mathbb{Z}_3$

${}_{\mathbb{Z}}\mathrm{Hom}(\mathbb{Z}_3, \mathbb{Z}_2 \oplus \mathbb{Z}_3) \cong \mathbb{Z}_3$

$\underline{\mathrm{Hom}(M \oplus N, L) \cong \mathrm{Hom}(M, L) \oplus \mathrm{Hom}(N, L)}$

$$* = \begin{bmatrix} \mathbb{Z}_6 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix}$$

Boljši način:

$$\mathbb{Z}_6 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_2 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_3)$$

$$\Rightarrow \mathrm{End}(\mathbb{Z}_6 \oplus \mathbb{Z}_3) = \mathrm{End}(\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3) = \begin{bmatrix} \mathrm{End}(\mathbb{Z}_2) & \mathrm{Hom}(\mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_2) \\ \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_3 \oplus \mathbb{Z}_3) & \mathrm{End}(\mathbb{Z}_3 \oplus \mathbb{Z}_3) \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix} \end{bmatrix} \cong \mathbb{Z}_2 \times M_2(\mathbb{Z}_3)$$

BIMODULI

${}_K M \; \dots \; M_{K^{op}}$

$K^{op} = \{ x^{op} ; x \in K \}$

$x^{op} \cdot y^{op} = (yx)^{op}$

Grupe: $G \cong G^{op}$, $x \mapsto x^{-1}$

Definicija: Bimodul ${}_K M_S$

K, S kolobarja

${}_K M$ levi K-modul
M_S desni S-modul

$$(\underset{\substack{\cap \\ K}}{x}\cdot m)\cdot \underset{\substack{\cap \\ S}}{y} = x\cdot(m\cdot y)$$

Primer: ${}_K K_K$ ... bimodul

${}_K M_{K^{op}}$ ... ni bimodul (razen K komutativen)

${}_K M_{End(M)^{op}}$ ... bimodul : $(xm)\cdot\varphi^{op} = \varphi(xm) =$

$= \varphi(xm) = x\cdot\varphi(m) = x(m\cdot\varphi^{op})$

${}_{End(M_{End(M)^{op}})} M_{End(M)^{op}}$ ... bimodul

$K\text{-Mod} \longrightarrow \text{Mod-}K^{op}$

---

2) Naj bosta ${}_K M_S$, ${}_K N_R$ bimodula, K, R, S kolobarji.
Premisli, da ima $Hom({}_K M, {}_K N)$ naravno strukturo (S,R)-bimodula.

(Ni K-modul!)

$s \in S$, $\varphi \in Hom(M,N)$: $s\varphi := (m \mapsto \varphi(ms))$

$r \in R$, $\varphi \in Hom(M,N)$: $\varphi r := (m \mapsto \varphi(m)r)$

$(\varphi+\psi)(m) = \varphi(m) + \psi(m)$

$(t\cdot\varphi)(m) = \varphi(m\cdot t)$

$\underline{Hom(M,N)}$ $\underline{\text{levi } S\text{-modul}}$

$(s+t)\varphi = (m \mapsto \varphi(m(s+t)))$

$= (m \mapsto \varphi(ms+mt))$

$$= (m \mapsto \varphi(ms)) + (m \mapsto \varphi(mt))$$

$$= s\varphi + t\varphi$$

$\varphi(s+t) = s\varphi + t\varphi$ ✓

$1 \cdot \varphi = \varphi$ ✓

$$[(st)\varphi](m) = \varphi(m(st))$$
$$= \varphi((ms)t)$$
$$= (t\varphi)(ms)$$
$$= (s\cdot(t\varphi))(m)$$

$t\cdot\varphi \in \mathrm{Hom}(M,N) \iff (t\cdot\varphi)(x\cdot m) = x((t\cdot\varphi)(m))$

$$(t\cdot\varphi)(x\cdot m) = \varphi((xm)t)$$

bimodul

$$= \varphi(x(mt))$$
$$= x\varphi(mt)$$

$$(s\cdot(\varphi r))(m) = (\varphi r)(ms) = \varphi(ms)\, r$$
$$\| $$
$$((s\varphi)\cdot r)(m) = (s\varphi)(m)\, r = (\varphi(ms))\, r$$

DUALI

${}_K M$ levi K-modul

Definicija: Dual $M^*_K = \mathrm{Hom}({}_KM, {}_KK_K)$

Dual M^*_K postane desni K-modul, ker je kodomena bimodul.

---

3) Določi dual modula ${}_{M_2(F)}M := {}_{M_2(F)}\begin{bmatrix} F \\ F \end{bmatrix}$.

$({}_{M_2(F)}M)^* = M^*_{M_2(F)} = \mathrm{Hom}({}_{M_2(F)}M, {}_{M_2(F)}M_2(F)) = *$

$\begin{bmatrix} F & F \\ F & F \end{bmatrix} \cong \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix} \cong M \oplus M$

$* = \mathrm{Hom}(M, M \oplus M) = \mathrm{End}(M) \overset{\text{obseg}}{\oplus} \mathrm{End}(M) \cong F \oplus F$

$F \cong \mathrm{Hom}(M,M) = \mathrm{End}(M)$

$(\supseteq)$ $\varphi \in \mathrm{Hom}_K(M,M) \leq \mathrm{Hom}_F(M,M) \cong M_2(F)$

$\varphi(Ax) = A\varphi(x) \quad \forall A, \forall x$

matrika φ

$BAx = ABx$

$\Rightarrow AB = BA \quad \forall A \in K$

$\Rightarrow B \in Z(M_2(F)) = F \cdot I$

$(\subseteq)$ Očitno

$F \oplus F$ ima strukturo desnega modula

$\varphi \in \mathrm{Hom}(M,K) \xrightarrow{\cong} [\varphi_1, \varphi_2]$

$$\left(\varphi(m) = (\varphi_1(m), \varphi_2(m)) = \left(\varphi(m)\begin{bmatrix}1\\0\end{bmatrix}, \varphi(m)\begin{bmatrix}0\\1\end{bmatrix}\right)\right)$$

$$\varphi(m) = \left[\begin{array}{c|c} \varphi_1(m) & \varphi_2(m) \end{array}\right]$$

$$\varphi(Am) = [\varphi_1(Am) \,\vdots\, \varphi_2(Am)]$$

$$\parallel$$

$$A\varphi(m) = [A\varphi_1(m) \,\vdots\, A\varphi_2(m)]$$

$$\varphi \cdot A \overset{M^*}{=} (m \mapsto \varphi(m)A)$$

$$[\varphi_1, \varphi_2] \cdot \underset{\begin{bmatrix}a & c\\ b & d\end{bmatrix}}{\underset{\parallel}{A}} = [\varphi_1 a + \varphi_2 b, \varphi_1 c + \varphi_2 d]$$

Hočemo pokazati, da je M^* izomorfna vrsticam.

$$A = \begin{bmatrix}a & c\\ b & d\end{bmatrix}, \quad m = \begin{bmatrix}x\\ y\end{bmatrix}$$

$$\varphi(m) = [\varphi_1(m), \varphi_2(m)] =$$

$$= [\lambda_1 m, \lambda_2 m] =$$

$$= \begin{bmatrix}\lambda_1 x & \lambda_2 x\\ \lambda_1 y & \lambda_2 y\end{bmatrix}$$

$$\varphi(m) \cdot A = \begin{bmatrix}\lambda_1 x & \lambda_2 x\\ \lambda_1 y & \lambda_2 y\end{bmatrix}$$

$$= \begin{bmatrix}\lambda_1 ax + \lambda_2 bx & \lambda_1 cx + \lambda_2 dx\\ \lambda_1 ay + \lambda_2 by & \lambda_1 cy + \lambda_2 dy\end{bmatrix} =$$

$$= \begin{bmatrix}(\lambda_1 a + \lambda_2 b)x & (\lambda_1 c + \lambda_2 d)x\\ (\lambda_1 a + \lambda_2 b)x & (\lambda_1 c + \lambda_2 d)y\end{bmatrix}$$

$$\varphi \cdot A = [\lambda_1 a + \lambda_2 b, \lambda_1 c + \lambda_2 d] = [\varphi_1 a + \varphi_2 b, \varphi_1 c + \varphi_2 d] \qquad \checkmark$$

17.11.

1) Določi $({}_{\mathbb{Z}}\mathbb{Z}_n)^*$.

$({}_{\mathbb{Z}}\mathbb{Z}_n)^* = \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z})$

$1 + n\mathbb{Z} \overset{?}{\mapsto} 1$ ne gre, ker $\underset{\underset{\varphi(0)=0}{\|}}{\varphi(n)} = n\varphi(1) = n \cdot 1 = n$:(

Tudi ostale neničelne ne gredo ...

$\Rightarrow ({}_{\mathbb{Z}}\mathbb{Z}_n)^* = 0$

---

2) K cel kolobar (komutativen brez deliteljev niča)
${}_K M$ K-modul

$\mathrm{tor}(M) := \{ m \in M ; \ x \cdot m = 0 \text{ za nek } x \in K \setminus \{0\} \}$

Dokaži, da je $\mathrm{tor}(M) \leq M$ in za vsak $\varphi : M \to N$ velja $\varphi(\mathrm{tor}(M)) \subseteq \mathrm{tor}(N)$.

$\mathrm{tor}(M) \leq {}_K M$

Vsota

$m \in \mathrm{tor} M : \ \overset{0}{\underset{\ }{\overset{\#}{x}}} \cdot m = 0$
$n \in \mathrm{tor} M : \ \underset{0}{\underset{\#}{y}} \cdot n = 0$

$\Rightarrow xy(m-n) = xym - xyn = y(\overset{0}{\overset{\#}{x m}}) - x(\overset{0}{\overset{\#}{y n}}) = 0$

Množenje s skalarjem

$m \in \operatorname{tor} M : \overset{\overset{0}{\neq}}{x} \cdot m = 0$
$y \in K$

$\Rightarrow x(ym) = y(xm) = 0$

$\varphi(\operatorname{tor}(M)) \subseteq \operatorname{tor}(N)$

$m \in \operatorname{tor} M : xm = 0$

$\underset{\parallel}{\varphi}(xm) = \varphi(0) = 0$

$x\varphi(m) = xn$

$\Rightarrow n = \varphi(m) \in \operatorname{tor} N$

$\operatorname{tor}(M/\operatorname{tor} M) = 0$

M prost: $M \cong K^{(I)} = \bigoplus_{i \in I} K$

---

3) Dokaži: M prost $\Rightarrow \operatorname{tor} M = 0$

$\varphi : M \xrightarrow{\cong} \bigoplus_{i \in I} K$

$\varphi(\operatorname{tor} M) \subseteq \operatorname{tor}(\bigoplus_{i \in I} K)$

φ injektiven

Dovolj: $tor(\bigoplus_{i\in I} K) = 0$

$\bigoplus_{i\in I} K = \{(x_i)_{i\in I} \mid x_i \in K,\ x_i = 0$ razen za končno mnogo $i \in I\}$

$y \in K \setminus \{0\}$

$(x_i)_{i\in I} \in tor(\bigoplus_{i\in I} K)$

$y \cdot (x_i)_{i\in I} = (y \cdot x_i)_{i\in I} = 0 \iff y \cdot x_i = 0 \ \forall i$

$\overset{K \text{ cel}}{\Rightarrow} x_i = 0 \ \forall i$

---

4) Ali so $\mathbb{Z}_n$ prosti $\mathbb{Z}$-moduli?

Ne, ker imajo torzijo.

Izrek: Naj bo M končno-generiran $\mathbb{Z}$ modul. Potem velja:
M prost $\iff$ M torzijsko prost

---

5) Pokaži, da $_{\mathbb{Z}}\mathbb{Q}$ ni prost modul.

$\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$:

$(cb) \cdot \frac{a}{b} - (ad) \cdot \frac{c}{d} = 0$

Vsaka dva elementa sta linearno odvisna.

$\Rightarrow$ Ne moremo imeti baze z več kot enim elementom.

Lahko je mogoče izomorfen $\mathbb{Z}$ kot $\mathbb{Z}$-modul.

$$\Rightarrow \mathbb{Q} = \left(\tfrac{a}{b}\right) = \left\{ x \cdot \tfrac{a}{b} \,;\, x \in \mathbb{Z} \right\}$$

To pa očitno ni res.

24.11.

EKSAKTNA ZAPOREDJA

$$0 \xrightarrow{0} L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{0} 0$$

$\operatorname{im} 0 = \ker \varphi$ (φ injektiven)
$\operatorname{im} \varphi = \ker \psi$
$\operatorname{im} \psi = \ker 0$ (ψ surjektiven)

$(M/\varphi(L) \cong N)$

$$0 \to \mathbb{Z} \xrightarrow[x \mapsto 2x]{\varphi} \mathbb{Z} \xrightarrow[y \mapsto y \bmod 2]{\psi} \mathbb{Z}_2 \to 0$$

$n \mapsto 2n \mapsto 0 \Rightarrow \operatorname{im} \varphi \subseteq \ker \psi$
$\ker \psi \subseteq \operatorname{im} \varphi$

$$0 \to \mathbb{Z} \xrightarrow[n \mapsto n]{} \mathbb{R} \xrightarrow[x \mapsto e^{2\pi i x}]{} \mathbb{T} \to 0$$

DN)

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\lambda"] & M \arrow[r, "\psi"] \arrow[d, "\sigma"] & N \arrow[r] \arrow[d, "\nu"] & 0 \\ 0 \arrow[r] & L' \arrow[r] & M' \arrow[r, "\psi'"] & N' \arrow[r] & 0 \end{tikzcd}$$

$m \mapsto n$, $\ell \mapsto \tilde{m}$, $\ell' \mapsto \tilde{m}'$, $m' \mapsto n'$, $m'' \mapsto n''$, $(m''-m') = \tilde{m}'$

a) λ, ν injektivni $\Rightarrow$ σ injektivna

Vaje list

b) λ, ν surjektivni $\Rightarrow$ σ surjektivna

Komutativnost diagrama $\Rightarrow n' = n''$

$m' - m'' \in \ker \psi$

$m' = m'' - \tilde{n}'$

$\Rightarrow m - \tilde{m} \mapsto m'' - \tilde{m}' = m'$

RAZPADNA EKSAKTNA ZAPOREDJA

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "id"] & L \oplus N \arrow[r] \arrow[d, "\sigma"] & N \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

$\ell \mapsto (\ell, 0)$, $(\ell, n) \mapsto n$

Zaporedje razpadno, če je ekvivalentno $0 \to L \to L \oplus N \to N \to 0$.

P projektiven modul, če je vsako eksaktno zaporedje $0 \to L \to M \to P \to 0$ razpadno.

Ekvivalentno:

i)
$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists g"'] \arrow[d, "\forall \varphi"] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$
eksaktno

ii) P direktni sumand prostega modula:

$$\exists N: \ P \oplus N \cong K^{(I)} = \bigoplus_{i \in I} K$$

Velja: Prost $\Rightarrow$ Projektiven

---

1) Dokaži, da je $\begin{bmatrix} F \\ F \end{bmatrix}$ projektiven $M_2(F)$-modul, ki ni prost.

$N = \begin{bmatrix} F \\ F \end{bmatrix}$

prost za $I = \{1\}$

$$\underbrace{\begin{bmatrix} F \\ F \end{bmatrix} \oplus \begin{bmatrix} F \\ F \end{bmatrix}}_{\text{zunanja}} = M_2 F = \underbrace{\begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}}_{\text{notranja}}$$

$\Rightarrow N$ projektiven

Denimo, da je $\begin{bmatrix} F \\ F \end{bmatrix}$ prost $M_2(F)$ modul:

$$\underbrace{\begin{bmatrix} F \\ F \end{bmatrix}}_{\dim = 2} \cong \underbrace{M_2(F) \oplus M_2(F)}_{\dim = 4 > 2}$$

$\times$

---

2) Dokaži: P projektiven $\Rightarrow$ P torzijsko prost (K cel)

$P \xrightarrow{\varphi} M$

$\varphi(tor(P)) \subseteq tor(M)$

$$\begin{array}{c} P \oplus N \cong K^{(I)} \\ \uparrow \\ P \longrightarrow K^{(I)} \end{array}$$

$$\varphi(tor(P)) \subseteq tor(K^{(I)}) = 0$$

φ injektiven $\Rightarrow$ $tor(P) = 0$

---

prost $\Rightarrow$ projektiven $\overset{K \text{ cel}}{\Rightarrow}$ torzijsko prost

K glavni
P konč. gen.

Kategorija $\underline{K\text{-Mod}}$:

Objekti: K-moduli
Morfizmi: homomorfizmi K-modulov

$$\underset{\text{funktor}}{Hom({}_K P, \cdot)}: \underline{K\text{-Mod}} \longrightarrow \underline{Ab}$$
$$_K M \longmapsto Hom(P, M)$$

$$(M \xrightarrow{f} N) \longmapsto \begin{pmatrix} Hom(P,M) \longrightarrow Hom(P,N) \\ (P \xrightarrow{\varphi} M) \longmapsto f \circ \varphi \end{pmatrix}$$

---

3) Dokaži: $Hom(P, \cdot): \underline{K\text{-Mod}} \longrightarrow \underline{Ab}$ je funktor

- Ohranja identiteto:

$$(M \xrightarrow{id} M) \longmapsto (\underset{id \circ \sqcup = \sqcup}{Hom(P,M) \longrightarrow Hom(P,M)})$$

✓

- Ohranja kompozitum:

$$\begin{tikzcd}
(M \xrightarrow{f} N \xrightarrow{g} L) \arrow[r, mapsto, "\mathrm{Hom}(P,\cdot)"] \arrow[d, mapsto, "\text{komp.}"'] & (\mathrm{Hom}(P,M) \xrightarrow{f\circ_} \mathrm{Hom}(P,N) \xrightarrow{g\circ_} \mathrm{Hom}(P,L)) \arrow[d, mapsto, "\text{komp.}"] \\
g\circ f \arrow[r, mapsto, "\mathrm{Hom}(P,\cdot)"] & (g\circ f)\circ_ = g\circ(f\circ_)
\end{tikzcd}$$

✓

$\mathrm{Hom}(P,\cdot)$ eksakten:

$0 \to L \xrightarrow{\varphi} M \xrightarrow{f} N \to 0$ eksaktno

$\Rightarrow 0 \to \mathrm{Hom}(P,L) \xrightarrow{\varphi\circ\_} \mathrm{Hom}(P,M) \xrightarrow{f\circ\_} \mathrm{Hom}(P,N) \to 0$ eksaktno

---

4) Dokaži: P projektiven $\Leftrightarrow$ $\mathrm{Hom}(P,\cdot)$ eksakten

$(\Rightarrow)$ i) $P = K$:

$$\mathrm{Hom}(K, {}_KM) \xrightarrow{\cong} {}_KM$$
$$f \mapsto f(1)$$

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "\varphi"] \arrow[d, phantom, "\cong" rotate=90] & M \arrow[r, "f"] \arrow[d, phantom, "\cong" rotate=90] & N \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
0 \arrow[r] & \mathrm{Hom}(P,L) \arrow[r, "\varphi\circ_"] & \mathrm{Hom}(P,M) \arrow[r, "f\circ_"] & \mathrm{Hom}(P,N) \arrow[r] & 0
\end{tikzcd}$$

Velja ✓

ii) P prost:

$$P \cong K^{(I)}$$

$$\mathrm{Hom}(P,\cdot) = \mathrm{Hom}(\bigoplus_I K, \cdot) = \bigoplus_I \mathrm{Hom}(K,\cdot)$$

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "\varphi"] \arrow[d, phantom, "\cong" rotate=90] & M \arrow[r, "f"] \arrow[d, phantom, "\cong" rotate=90] & N \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
0 \arrow[r] & \oplus\mathrm{Hom}(K,L) \arrow[r, "\varphi\circ_"] & \oplus\mathrm{Hom}(K,M) \arrow[r, "f\circ_"] & \oplus\mathrm{Hom}(K,N) \arrow[r] & 0
\end{tikzcd}$$

$$(\ell_1, \ell_2, \dots) \longmapsto (f(\ell_1), f(\ell_2), \dots)$$

Opomba: $\operatorname{Hom}(\bigoplus_{i\in I} M_i, N) = \prod_{i\in I} \operatorname{Hom}(M_i, N)$

iii) $P \oplus Q \cong K^{(I)}$:

$$\operatorname{Hom}(P\oplus Q, M) \xrightarrow{\varphi \circ \_} \operatorname{Hom}(P\oplus Q, N) \longrightarrow 0$$

$$f \oplus g \longmapsto \underbrace{(\varphi\circ f)}_{= h} \oplus \underbrace{(\varphi \circ g)}_{= 0}$$

$f: P \to M$
$g: Q \to M$

$h: P \to N$

$\dashrightarrow f \mapsto \varphi \circ f$

8.12.

TENZORSKI PRODUKT

K komutativen

${}_K M$, ${}_K N$ modula

$$M \otimes_K N = \{\sum_{i=1}^{\ell} m_i \otimes n_i \ ;\ m_i \in M, n_i \in N, \ell \in \mathbb{N}_0\}$$

Bilinearnost:

$$(m + m') \otimes n = m \otimes n + m' \otimes n$$

$$m \otimes (n + n') = m \otimes n + m \otimes n'$$

Modulska struktura:

$$x \cdot \sum m_i \otimes n_i = \sum x \cdot (m_i \otimes n_i) = \sum (x m_i) \otimes n_i$$

$(xm) \otimes n = m \otimes (xn)$

Univerzalna lastnost tenzorskega produkta:

$$\begin{tikzcd}
{}_kM \times {}_kN \arrow[r, "\text{bilin.}", "\varphi"'] \arrow[d, "(m,n) \mapsto m\otimes n"'] & L \\
M \otimes N \arrow[ur, dashed, "\exists! \overline{\varphi}"'] &
\end{tikzcd}$$

---

1) Zapiši $\left(\frac{1}{2}, \frac{1}{2}\right) \otimes (1,1) + \left(\frac{1}{2}, 1\right) \otimes (0,1) + (-1,0) \otimes \left(\frac{1}{2}, 1\right) \in \mathbb{R}^2 \otimes \mathbb{R}^2$ kot enostavni tenzor $u \otimes v$.

Razvijemo druge komponente po bazi:

$$\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \otimes (1,1) + \left(\tfrac{1}{2}, 1\right) \otimes (0,1) + (-1,0) \otimes \left(\tfrac{1}{2}, 1\right) =$$

$$= \left(\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \otimes (1,0) + \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \otimes (0,1)\right) + \left(\left(\tfrac{1}{2}, 1\right) \otimes (0,1)\right) + \left(\tfrac{1}{2}(-1,0) \otimes (1,0) + (-1,0) \otimes (0,1)\right) =$$

$$= \left(\left(\tfrac{1}{2}, \tfrac{1}{2}\right) + \left(\tfrac{1}{2}, 1\right) + (-1,0)\right) \otimes (0,1) + \left(\left(\tfrac{1}{2}, \tfrac{1}{2}\right) + \tfrac{1}{2}(-1,0)\right) \otimes (1,0) =$$

$$= \left(0, \tfrac{3}{2}\right) \otimes (0,1) + \left(0, \tfrac{1}{2}\right) \otimes (1,0) =$$

$$= 3\left(0, \tfrac{1}{2}\right) \otimes (0,1) + \left(0, \tfrac{1}{2}\right) \otimes (1,0) =$$

$$= \left(0, \tfrac{1}{2}\right) \otimes \left((0,3) + (1,0)\right) =$$

$$= \left(0, \tfrac{1}{2}\right) \otimes (1,3)$$

---

2) Poišči primer vek. prostorov U, V in tenzorja $x \in U \otimes V$, ki ni enostavni.

Če je $U = V = \mathbb{F}$, so vsi tenzorji enostavni:

$$\sum x_i \otimes y_i = \sum x_i (1 \otimes y_i) = \sum x_i y_i (1 \otimes 1) = \left(\sum x_i y_i\right) \otimes 1$$

Opomba: $M \otimes_K K \cong M$

Poglejmo zdaj $U = V = \mathbb{R}^2$:

$$(1,0) \otimes (0,1) + (0,1) \otimes (1,0) = x \otimes y$$

$x = (x_1, x_2)$
$y = (y_1, y_2)$

$$(x_1, x_2) \otimes (y_1, y_2) = (x_1(1,0) + x_2(0,1)) \otimes (y_1(1,0) + y_2(0,1)) =$$

$$= (x_1 y_1)(1,0) \otimes (0,1) + (x_1 y_2)(1,0) \otimes (0,1) + (x_2 y_1)(0,1) \otimes (1,0) + (x_2 y_2)(0,1) \otimes (0,1)$$

$x_1 y_1 = 0$
$x_1 y_2 = 1$
$x_2 y_1 = 1$
$x_2 y_2 = 0$

$\Rightarrow x_1 x_2 y_1 y_2 = 0$
$\quad x_1 x_2 y_1 y_2 = 1$

$\times$

15.12.

1) Dokaži $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}$.

$$\begin{tikzcd} \mathbb{Q} \times \mathbb{R} \arrow[r, "\varphi \text{ bilin.}"] \arrow[d] & \mathbb{R} \\ \mathbb{Q} \otimes \mathbb{R} \arrow[ur, dashed, "\exists! \,\bar\varphi \text{ homo.}"'] & \end{tikzcd}$$

$$\varphi : \mathbb{Q} \times \mathbb{R} \xrightarrow{\varphi} \mathbb{R} \quad \text{bilinearna}$$
$$(q, r) \longmapsto q \cdot r$$

$$\bar\varphi\left(\sum q_i \otimes r_i\right) = \sum \bar\varphi(q_i \otimes r_i) = \sum \varphi(q_i, r_i)$$

dobra definiranost ✓

$$(p+q, r) \overset{\varphi}{\longmapsto} (p+q) r = pr + qr = \varphi(p, r) + \varphi(q, r)$$

$\bar\varphi$ očitno surjektivna ✓

$\bar\varphi$ injektivna ?

$$\psi : \mathbb{R} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R}$$
$$r \longmapsto 1 \otimes r$$

$$\psi(r+s) = 1 \otimes (r+s) = 1 \otimes r + 1 \otimes s = \psi(r) + \psi(s)$$

ψ aditivna ✓

dobra definiranost ✓

$$\bar\varphi(\psi(r)) = \bar\varphi(1 \otimes r) = r \quad ✓$$

$$\psi\left(\bar\varphi\left(\sum q_i \otimes r_i\right)\right) = \sum \psi(\bar\varphi(q_i \otimes r_i)) = \sum \psi(q_i r_i) = \sum 1 \otimes q_i r_i =$$

$$= \sum 1 \otimes \frac{a_i}{b_i} r_i = \sum a_i \otimes \frac{r_i}{b_i} = \sum \frac{b_i a_i}{b_i} \otimes \frac{r_i}{b_i} = \sum \frac{a_i}{b_i} \otimes \frac{b_i}{b_i} r_i =$$

$$= \sum q_i \otimes r_i \quad ✓$$

$R \subseteq K$

${}_K M \rightsquigarrow {}_R M$ restrikcija skalarjev

${}_R M \rightsquigarrow {}_K K \otimes_R M$ razširitev skalarjev $\quad \left(\lambda \sum \mu_i \otimes m_i := \sum \lambda \mu_i \otimes m_i\right)$

Primer: ${}_{\mathbb{Z}}\mathbb{R} = {}_{\mathbb{Z}}({}_{\mathbb{Q}}\mathbb{R})$ restrikcija $\mathbb{Q} \rightsquigarrow \mathbb{Z}$

$\mathbb{Q} \otimes \mathbb{R} = {}_{\mathbb{Q}}\mathbb{R}$ razširitev $\mathbb{Z} \rightsquigarrow \mathbb{Q}$

Primer:

$$
\begin{tikzcd}
{}_{\mathbb{Z}}\mathbb{Z} \arrow[r, squiggly, "\text{razširitev}"] & {}_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}) \cong {}_{\mathbb{Q}}\mathbb{Q} \arrow[d, squiggly, "\text{restrikcija}"] \\
 & {}_{\mathbb{Z}}\mathbb{Q}
\end{tikzcd}
$$

Žal nista zares inverzna :(

Sta samo v eno smer ...

---

2) Naj bo $d = \gcd(n,m)$. Dokaži $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m \cong \mathbb{Z}_d$.

$\varphi(a,b) = a \cdot b$

$\varphi(a+kn, b) = ab + \cancelto{0}{knb}_{d \mid n} = ab \quad \checkmark$

$$
\begin{tikzcd}
\mathbb{Z}_n \times \mathbb{Z}_m \arrow[r, "\varphi"] \arrow[d] & \mathbb{Z}_d \\
\mathbb{Z}_n \otimes \mathbb{Z}_m \arrow[ur, dashed, "\exists! \Phi"'] &
\end{tikzcd}
$$

$$\sum a_i \otimes b_i \xmapsto{\varphi} \sum a_i b_i$$

$$\begin{aligned} \mathbb{Z}_d &\xrightarrow{\psi} \mathbb{Z}_n \otimes \mathbb{Z}_m \\ c &\longmapsto 1 \otimes c \end{aligned}$$

$$\psi(c+kd) = 1 \otimes (c+kd) = *$$

$$d = n \cdot a + m \cdot b$$

$$* = 1 \otimes c + 1 \otimes kd = 1 \otimes c + 1 \otimes k(na + \cancel{mb}^{\,0}) =$$

$$= 1 \otimes c + 1 \otimes kna = 1 \otimes c + \cancel{n \otimes ka}^{\,0} = 1 \otimes c \quad \checkmark$$

---

3) Pokaži $\mathrm{Hom}({}_K M \otimes_K N, {}_K L) \cong \mathrm{Hom}({}_K M, {}_K \mathrm{Hom}(N, L))$.

$$\begin{aligned} \Phi : \ & \mathrm{Hom}(M \otimes N, L) \longrightarrow \mathrm{Hom}(M, \mathrm{Hom}(N, L)) \\ & (\varphi : M \otimes N \to L) \longmapsto \psi : M \to \mathrm{Hom}(N, L) \\ & \qquad\qquad\qquad\qquad\qquad m \mapsto f : N \to L \\ & \qquad\qquad\qquad\qquad\qquad\qquad n \mapsto (m \otimes n) \end{aligned}$$

$$\begin{aligned} \Psi : \ & \mathrm{Hom}(M, \mathrm{Hom}(N, L)) \longrightarrow \mathrm{Hom}(M \otimes N, L) \\ & \begin{pmatrix} \psi : M \to \mathrm{Hom}(N, L) \\ m \mapsto \psi_m : N \to L \end{pmatrix} \longmapsto \begin{pmatrix} M \times N \longrightarrow L \\ (m, n) \mapsto \psi_m(n) \end{pmatrix} \\ & \qquad\qquad\qquad\qquad \rightsquigarrow M \otimes N \longrightarrow L \end{aligned}$$

$(m, n) \mapsto \psi_m(n)$ bilin.

$$\psi_{m+m'}(n) = \psi_m(n) + \psi_{m'}(n) \qquad (\psi_{m+m'} = \psi_m + \psi_{m'}, \ \psi \text{ homo.})$$

$$\psi_{\lambda m}(n) = \lambda \psi_m(n) = \psi_m(\lambda n) \qquad (\psi_{\lambda m} = \lambda \psi_m, \ \psi \text{ homo.})$$

$\Phi \circ \Psi = id$

$\psi : M \to Hom(N, L)$ poljuben homo.

$\Phi(\Psi(\psi)) \overset{?}{=} \psi$

$\Phi(\Psi(\psi))(m) \overset{?}{=} \psi(m)$

$(\Phi(\Psi(\psi))(m))(n) \overset{?}{=} \psi(m)(n)$

$$(\Phi(\underbrace{\Psi(\psi)}_{\varphi})(m))(n) \overset{\text{def. } \Phi}{=} \varphi(m \otimes n) = \underbrace{\Psi(\psi)}_{\varphi}(m \otimes n) \overset{\text{def. } \Psi}{=} \psi_m(n) = \psi(m)(n)$$

✓

Drugo smer podobno ...

Poseben primer: $K = \mathbb{Z}$, $N = \mathbb{Q}$

$$Hom_{\mathbb{Q}}(\underbrace{{}_{\mathbb{Q}}(M \otimes \mathbb{Q})}_{\text{restrikcija}}, {}_{\mathbb{Q}}L) \cong Hom_{\mathbb{Z}}({}_{\mathbb{Z}}M, \underbrace{{}_{\mathbb{Z}}Hom_{\mathbb{Q}}(\mathbb{Q}, L)}_{\text{restrikcija}})$$

${}_{\mathbb{Z}}Hom_{\mathbb{Q}}({}_{\mathbb{Q}}\mathbb{Q}_{\mathbb{Z}}, {}_{\mathbb{Q}}L) \cong {}_{\mathbb{Z}}L$ restrikcija

Funktorja razširitev in restrikcija sta adjungirana.

$\_\_ \otimes N$ je adjungiran $Hom(N, \_\_)$.

$$R\text{-mod} \overset{\text{ekv.}}{\longleftrightarrow} M_n(R)\text{-mod}$$

$$\underset{G}{\overset{F}{\rightleftarrows}}$$

$F \circ G \overset{\text{nar.}}{\cong} id$

Šibka ekvivalenca: Obstoj adj. funktorja

Iz te naloge sledi: $(\oplus M_i) \otimes N \cong \oplus (M_i \otimes N)$

V ... vektorski prostor nad F

$V \otimes_F V^* \xrightarrow{\varphi} \mathrm{End}(V)$

$\sum v_i \otimes \varphi_i \longmapsto (u \mapsto \sum \varphi_i(u) \cdot v_i)$

$\begin{bmatrix} \\ \\ \end{bmatrix} \otimes \begin{bmatrix} & & \end{bmatrix} \longmapsto \begin{bmatrix} \\ \\ \end{bmatrix} \begin{bmatrix} & & \end{bmatrix}$ matrika ranga 1

V končnodim $\Rightarrow$ φ izomorfizem

22.12.

ALGEBRE

K komutativen kolobar

A K-algebra:

i) A K-modul
ii) A kolobar
iii) $\lambda(xy) = (\lambda x) y = x(\lambda y)$

Primeri:

- $K \subseteq A$: A K-algebra $\overset{(iii)}{\Longleftrightarrow}$ $K \subseteq Z(A)$

- $K/I \in Z(A)$: $\lambda \cdot x = (\lambda + I)\cdot x \quad \forall \lambda \in K$
  $I \lhd K$

$A \otimes_K B$ ... običajen tenzorski produkt

$$\left(\sum (a \otimes b)\right) \cdot \left(\sum (a' \otimes b')\right) = \sum \sum aa' \otimes bb'$$

$$(a \otimes 1)\cdot(1 \otimes b) = a \otimes b = (1 \otimes b)(a \otimes 1)$$

$$\begin{tikzcd}
A \arrow[rr, "\text{homo } f"] \arrow[d, "a \,\mapsto\, a\otimes 1"'] & B \arrow[r, "\text{homo } g"] \arrow[dl, "b \,\mapsto\, b\otimes 1"] & C \\
A \otimes B \arrow[urr, bend right, "\exists!\, f\otimes g"', "a\otimes b \,\mapsto\, f(a)\cdot g(b)"] & &
\end{tikzcd}$$

$f(A)$, $g(B)$ komutirata : $f(a)\,g(b) = g(b)\,f(a)$

φ bilinearna ... $\varphi(\lambda a, b) = \varphi(a, \lambda b)$

$\Rightarrow \varphi(a,b) := f(a) \cdot f(b)$

---

1) K komutativen kolobar
   $p(x) \in K[x]$
   A K-algebra

   Dokaži: $A \otimes_K K[x]/(p) \cong A[x]/(p)$

   $$f : A \longrightarrow A[x]/(p)$$
   $$a \longmapsto a + (p)$$

$f(\lambda a) = \lambda a + (p) = \lambda(a + (p)) = \lambda f(a)$

$$g : K[X]/(p) \longrightarrow A[X]/(p)$$
$$\underset{\sum \lambda_i x^i}{g(x)} + (p) \longmapsto \underset{\sum(\lambda_i \cdot 1)x^i}{g(x)} + (p)$$

f, g homomorfizma ✓

$f(a) \cdot g(\sum \lambda_i x^i + (p)) = (a + (p)) \cdot (\sum(\lambda_i \cdot 1) x^i + (p)) =$

$= a \sum (\lambda_i \cdot 1) x^i + (p) = \sum (a \lambda_i \cdot 1) x^i + (p) =$

$= \sum (\lambda_i a \cdot 1) x^i + (p) = (\sum (\lambda_i \cdot 1) x^i) a + (p) =$

$= (\sum (\lambda_i \cdot 1) x^i + (p)) \cdot (a + (p)) = g(\sum \lambda_i x^i) f(a)$

$$A \otimes_K K[X]/(p) \longrightarrow A[X]/(p)$$
$$a \otimes (x^i + (p)) \longmapsto a x^i + (p)$$

$$\varphi : A[X]/(p) \longrightarrow A \otimes_K K[X]/(p)$$
$$\sum a_i x^i + (p) \longmapsto \sum a_i \otimes (x^i + (p))$$

φ dobro definirana ✓

---

2) Poišči izomorfizem $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\cong} \mathbb{C} \times \mathbb{C}$.

$z \otimes w \mapsto (z, w)$ ne deluje:

$\lambda(z \otimes w) = \lambda z \otimes w = z \otimes \lambda w$
$\lambda(z, w) = (\lambda z, \lambda w)$
$/\!/$

$z \otimes w \mapsto (zw, 1)$ ne deluje:

ni surjektiven

ni linearen v 1. faktorju: $(z+z') \otimes w \mapsto ((z+z')w, 1) = (zw, 1) + (z'w, 0)$

Vemo: $\mathbb{C} \cong \mathbb{R}[X]/(1+X^2)$

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[X]/(1+X^2) \longrightarrow \mathbb{C}[X]/(1+X^2)$$

$$1+X^2 = (X+i)(X-i)$$

$$\mathbb{C}[X]/(X+i)(X-i) \overset{CRT}{\cong} \underbrace{\mathbb{C}[X]/(X+i)}_{\cong \mathbb{C}} \times \underbrace{\mathbb{C}[X]/(X-i)}_{\cong \mathbb{C}} \cong \mathbb{C} \times \mathbb{C}$$

$$\Rightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[X]/(1+X^2) \longrightarrow \mathbb{C}[X]/(1+X^2)$$
$$z \otimes (a+bi) \longmapsto z \otimes (a+bX) \longmapsto z(a+bX)$$

$$\longrightarrow \mathbb{C}[X]/(X+i) \times \mathbb{C}[X]/(X-i) \longrightarrow \mathbb{C} \times \mathbb{C}$$
$$\longmapsto (z(a+bX), z(a+bX)) \longmapsto (z(a-bi), z(a+bi))$$

Opomba: $K/IJ \to K/I \times K/J$
$$x \longmapsto (x+I, x+J)$$

$$\Rightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\cong} \mathbb{C} \times \mathbb{C}$$
$$z \otimes w \longmapsto (z\overline{w}, z \cdot w)$$

---

3) Dokaži: $M_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{C})$

Vemo: $M_n(K) \otimes_K A \cong M_n(A)$ $(*)$

$$M_n(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{H} \overset{*}{\cong} (\underbrace{M_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}}) \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{R}) \otimes_{\mathbb{R}} (\underbrace{\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}}) \overset{*}{\cong} M_2(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H})$$

$\Rightarrow$ Dokazujemo: $M_2(\mathbb{C}) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$

Iz tega bo sledilo: $M_4(\mathbb{C}) \cong M_2(M_2(\mathbb{C}))$

$$\begin{aligned} \mathbb{C} &\longrightarrow M_2(\mathbb{C}) \\ z &\longmapsto \begin{bmatrix} z & \\ & z \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbb{H} &\longrightarrow M_2(\mathbb{C}) \\ \alpha+\beta i+\gamma j+\delta k &\longmapsto \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \end{aligned}$$

$$M_2(\mathbb{C}) \cong \underset{\dim V = 2}{\mathrm{End}({}_{\mathbb{C}}V)} \cong \mathrm{End}({}_{\mathbb{C}}\mathbb{H})$$

Poskusimo levo množenje:

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathrm{End}({}_{\mathbb{C}}\mathbb{H}) \\ h &\longmapsto h \cdot \sqcup \end{aligned}$$

Ampak množenje s skalarji ne komutira: $h \cdot (\underset{\substack{\cap \\ \mathbb{C} \\ \text{(skalar)}}}{z} \cdot 1) \neq z \cdot (h \cdot 1)$ ☹

Vzemimo desno množenje s konjugiranjem:

$$\begin{aligned} \mathbb{H} &\xrightarrow{\phi} \mathrm{End}({}_{\mathbb{C}}\mathbb{H}) \\ h &\longmapsto \sqcup \cdot \overline{h} \end{aligned}$$

To pa je res homo. vek. prostorov.

$$\phi(h' \cdot h) = \sqcup \cdot \overline{(h \cdot h')}$$

$$\phi(h') \circ \phi(h) = \phi(h')(\sqcup \cdot \overline{h}) = \sqcup \cdot \overline{h} \cdot \overline{h'}$$

1) Naj bo K cel kolobar ter M, N prosta K-modula. Dokaži, da je $M \otimes_K N$ neničeln.

Razširitev skalarjev:

$$\underline{(M \otimes_K N) \otimes_K F = (M \otimes_K F) \otimes_F (N \otimes_K F)}$$

$$(M \otimes_K F) \otimes_F (N \otimes_K F) = (M \otimes_K F) \otimes_F (F \otimes_K N) = M \otimes_K (F \otimes_F (F \otimes_K N)) =$$

$$= M \otimes_K (\underbrace{(F \otimes_F F)}_{=F} \otimes_K N) = M \otimes_K (F \otimes_K N) = M \otimes_K (N \otimes_K F) = (M \otimes_K N) \otimes_K F$$

$$M_R \otimes_R {}_R N$$

$$\varphi : M \to M \otimes_K F$$
$$m \mapsto m \otimes 1$$

$$\ker \varphi = \operatorname{tor} M = \{0\}$$

$$M / \operatorname{tor} M \hookrightarrow M \otimes F$$

$$M \otimes_K F = \{0\} \Rightarrow \ker \varphi = M = \{0\} \Rightarrow \times$$

---

2) V končnodimenzionalen nad F

$$\psi : V \otimes V^* \to \operatorname{End} V \cong M_n(F)$$
$$\sum v_i \otimes f_i \mapsto (x \mapsto \sum f_i(x) v_i)$$

$$\psi = ?$$

$$\dim V = \dim V^* = n$$
$$\dim(\operatorname{End} V) = n^2$$

$$e_i^* = \begin{cases} e_i \mapsto 1 \\ e_j \mapsto 0 \ ; \ j \neq i \end{cases}$$

$$\sum v_i \otimes f_i = \sum \alpha_{ij} e_i \otimes e_j^* \rightsquigarrow ($$

$$e_i \otimes e_j^* \mapsto \left(e_k \mapsto e_j^*(e_k) e_i = \begin{cases} 0 \ ; \ k \neq j \\ e_i \ ; \ k = j \end{cases} \right) = \begin{array}{c|c} & j \\ \hline i & \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \end{array} = E_{ij}$$

$$\psi^{-1} : \sum \alpha_{ij} E_{ij} \longmapsto \sum \alpha_{ij} \, e_i \otimes e_j^*$$

---

3) Naj bosta a,b algebraični števili. Dokaži, kdaj je $\mathbb{Q}(a) \otimes_{\mathbb{Q}} \mathbb{Q}(b)$ polje.

$$\mathbb{Q}(a) \otimes_{\mathbb{Q}} \mathbb{Q}(b) \longrightarrow \mathbb{Q}(a,b) \longrightarrow 0$$

$$x \otimes y \mapsto x \cdot y$$

$$\begin{tikzcd} \mathbb{Q}(a) \arrow[r, "f", "\text{hom. algeber}"'] & \mathbb{Q}(a,b) \\ \mathbb{Q}(b) \arrow[ur, "g", "\text{hom. algeber}"'] & \end{tikzcd}$$

$$\rightsquigarrow \mathbb{Q}(a) \otimes \mathbb{Q}(b) \xrightarrow{f \otimes g} \mathbb{Q}(a,b)$$

$$f : \mathbb{Q}(a) \longrightarrow \mathbb{Q}(a,b)$$
$$x \longmapsto x$$

$$g : \mathbb{Q}(b) \longrightarrow \mathbb{Q}(a,b)$$
$$x \longmapsto x$$

Komutativnost ✓

f, g homomorfizma ✓

$\varphi = f \otimes g \quad (x \otimes y \mapsto x \cdot y)$

$\sum_{i,j} \alpha_{ij} a^i b^j = \varphi\left(\sum \alpha_{ij} a^i \otimes b^j\right)$

φ surjektivna ✓

Kdaj je φ injektivna?

$$\dim(\mathbb{Q}(a) \otimes \mathbb{Q}(b)) \overset{?}{=} \dim \mathbb{Q}(a,b)$$

$$\begin{array}{ccc} \dim(\mathbb{Q}(a) \otimes \mathbb{Q}(b)) & \overset{?}{=} & \dim \mathbb{Q}(a,b) \\ \| & & \| \\ \dim \mathbb{Q}(a) \cdot \dim \mathbb{Q}(b) & & [\mathbb{Q}(a,b) : \mathbb{Q}] \\ \| & & \\ [\mathbb{Q}(a) : \mathbb{Q}] \cdot [\mathbb{Q}(b) : \mathbb{Q}] & & \end{array}$$

to mora biti isto

Preslikave iz polja so vedno injektivne.

$(\ker \varphi \triangleleft F \Rightarrow \ker \varphi = \{0\})$

Alternativno:

$\mathbb{Q}(b) \cong \mathbb{Q}[b] \cong \mathbb{Q}[X]/(m_b(x))$

$\mathbb{Q}(a) \otimes_{\mathbb{Q}} \mathbb{Q}[X]/(m_b(x)) \cong \frac{\mathbb{Q}(a)[X]}{(m_b(x))}$ polje $\iff$ $m_b(x)$ nerazcepen nad $\mathbb{Q}(a)$

$\iff [\mathbb{Q}(a)(b) : \mathbb{Q}(a)] = [\mathbb{Q}(b) : \mathbb{Q}]$

---

4) Naj bo U vektorski prostor. Pokaži, da iz $u \otimes u + v \otimes v = u \otimes v + v \otimes u$ sledi $u = v$.

$u \otimes u + v \otimes v - u \otimes v - v \otimes u = 0$

$u \otimes (u-v) - v \otimes (u-v) = 0$

$(u-v) \otimes (u-v) = 0$

U je vektorski prostor.

Če $u-v \neq 0$, potem je $\{u-v, e_2, \dots\}$ baza U.

Torej je $(u-v) \otimes (u-v)$ bazni vektor za $U \otimes U$.

Torej je $u = v$.

12.1.

Izrek: K glavni, M končno generiran K-modul

$$M \cong K/(d_1) \oplus \dots \oplus K/(d_k)$$

$d_1 | \dots | d_k$
d_1 ni obrnljiv
$d_1, \dots, d_k$ enolični

Primer: $K = \mathbb{Z} \Rightarrow d_i \in \mathbb{Z}$

$${}_{\mathbb{Z}}M \cong \mathbb{Z}_{d_1} \oplus \dots \oplus \mathbb{Z}_{d_k} \cong \mathbb{Z}^s \oplus \underbrace{\mathbb{Z}_{d_1} \oplus \dots \oplus \mathbb{Z}_{d_\ell}}_{\substack{d_i > 0 \\ d_1 | \dots | d_\ell}}$$

Primer: $\underset{\substack{\cup\!\shortmid \\ F}}{K} = F[X]$

$X \cdot \underset{\substack{\cap \\ F^n}}{m} \in F_n$

$\rightsquigarrow X \cdot \_ \cdot F^n \xrightarrow{\text{lin.}} F^n$

Obratno: $A : F^n \xrightarrow{\text{lin.}} F^n \rightsquigarrow f(X) \cdot \underset{\substack{\cap \\ F^n}}{m} := f(\overset{\text{matrika}}{A})\, m$

$$\Rightarrow F^n \cong F[X]/(f_1(x)) \oplus \dots \oplus F[X]/(f_k(x))$$

$$m \in \mathbb{F}^n$$

$$A \cdot m := X \cdot m$$

$$f(X) = X^m + a_{n-1}X^{n-1} + \dots + a_0 \equiv 0 \text{ v kvocientu}$$

$$X^m \equiv -a_0 - a_1 X - \dots - a_{m-1} X^{m-2}$$

$$A = \begin{array}{cc} & \begin{array}{cccc} 1 & x & \cdots & x^{n-1} \end{array} \\ x^{n-2} & \begin{bmatrix} \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & 0 & & \vdots \\ & & 1 \ddots & 0 & -a_{n-2} \\ & & & \ddots \; 1 & -a_{n-1} \end{bmatrix} & & & \\ & \square & & \\ & & \ddots & \\ & & & \square \end{bmatrix} \end{array}$$

... Frobeniusova normalna forma / racionalna kanonična forma