1.1) Naj bo $A=[-1,1]\times[-1,1)$ in $g: \mathbb{R}^2 \to \mathbb{R}^2/A$ kvocientna preslikava. Ali so spodnje množice odprte oz. zaprte?

$$a \sim b \iff a=b \vee (a\in A \wedge b\in A)$$

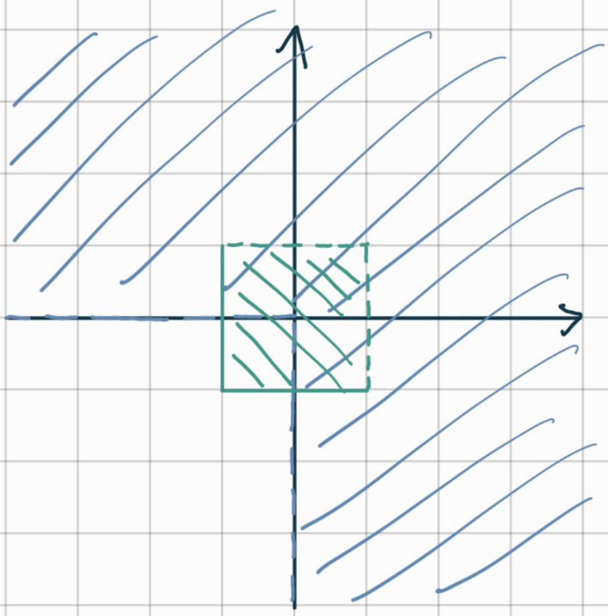
a) $g_*((-\infty,0]\times(-\infty,0])$

$$g^*(g_*((-\infty,0]\times(-\infty,0])) \overset{\text{nasičenje}}{=} (-\infty,0]\times(-\infty,0] \cup A$$

$\Rightarrow$ To ni odprto/zaprto v $\mathbb{R}^2$.

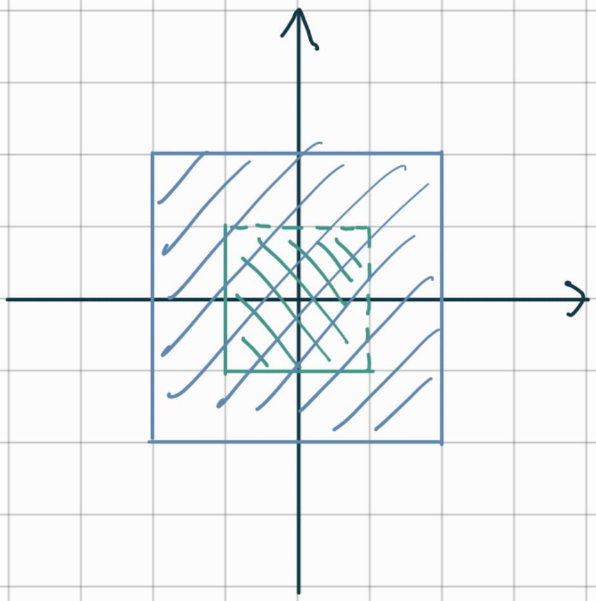
$\Rightarrow$ Množica ni odprta/zaprta v $\mathbb{R}^2/A$.

b) $g_*(\mathbb{R}^2\setminus(-\infty,0]\times(-\infty,0])$



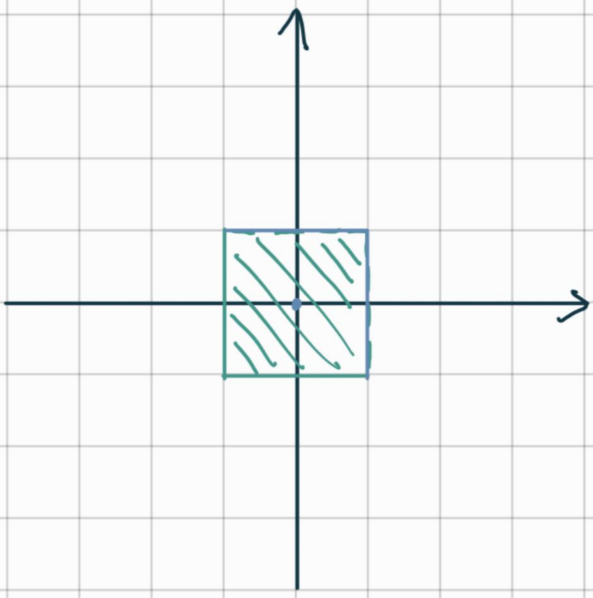
Nasičenje spet ni odprto/zaprto v $\mathbb{R}^2$.

c) $g_*([-2,2]\times[-2,2])$



Je zaprta in ni odprta.

d) $q_*([-1,1]\times\{1\} \cup \{1\}\times[-1,1] \cup \{(0,0)\})$



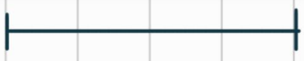
Je zaprta in ni odprta.

1.4) Poišči homeomorfen prostor.

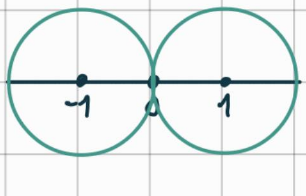
b) $[-1,1]/\{-1,0,1\} =: X$

$Y := \{(x,y)\in\mathbb{R}^2 \,;\, (|x|-1)^2+y^2=1\}$

X:



Y:



$$f: X \longrightarrow Y$$
$$x \longmapsto \begin{cases} (\cos(2\pi x)-1, \ \sin(2\pi x)) & ; \ x \in [-1,0] \\ (-\cos(2\pi x)+1, \ \sin(2\pi x)) & ; \ x \in [0,1] \end{cases}$$

f je zvezna.

f je očitno konstantna na A.

$\Rightarrow f$ je konstantna na ekvivalenčnih razredih.

$f(x) = (0,0) \Leftrightarrow x \in A$

f je injektivna na $[-1,1] \setminus A$

f je očitno surjektivna.

f slika iz kompakta v T_2 prostor.

$\Rightarrow f$ je kvocientna v ožjem smislu.

c) $\mathbb{R}^n / \overline{K(0,1)} =: X$

$Y := \mathbb{R}^n$

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\vec{x} \longmapsto \begin{cases} \vec{0} & ; \ \|\vec{x}\| < 1 \\ \vec{x} - \frac{\vec{x}}{\|\vec{x}\|_2} & ; \ \|\vec{x}\| > 1 \end{cases}$$

f je zvezna.

Očitno je f konstantna na A.

$f(\vec{x}) = \vec{0} \Leftrightarrow \vec{x} \in A$

$f|_{\mathbb{R}^n\setminus A} : \mathbb{R}^n\setminus A \longrightarrow \mathbb{R}^n\setminus\{\vec{0}\}$ je bijekcija.

$\Rightarrow$ f dela iste identifikacije in je surjektivna.

$\Rightarrow$ f je kvocientna v ožjem smislu.

$$\bigcup_{i\in I} K_i = X$$

$$Z^{zap} \subseteq X$$

$$f_*(Z) = f_*(Z\cap X) = f_*\Big(Z\cap \bigcup_{i\in I} K_i\Big) = f_*\Big(\bigcup_{i\in I} Z\cap K_i\Big) = \bigcup_{i\in I} f_*(Z\cap K_i)^{zap}$$

$$\left(\{\vec{a}\in\mathbb{R}^n \,;\, n-1 \le \|\vec{a}\| \le n\}\right)_{n\in\mathbb{N}}$$

d) $\mathbb{R}^n/\sim$, $\quad x\sim y :\Leftrightarrow \|x\| = \|y\|$

$\underline{\mathbb{R}^n/\sim \;\cong [0,\infty)}$, $n\geq 1$

$$\begin{aligned} f : \mathbb{R}^n &\longrightarrow [0,\infty) \\ \vec{x} &\longmapsto \|\vec{x}\| \end{aligned}$$

$t\in[0,\infty) : f((t,0,\dots,0)) = t$ ✓

$[x] = [y] \Leftrightarrow f(x) = f(y)$ ✓

$$\begin{tikzcd} X \arrow[r, bend left, "r"] & Y \arrow[l, bend left, "s"] \end{tikzcd}, \quad r\circ s = id_Y$$

$p : X\to X$
$p = s\circ r$

$S\subseteq Y$, $r^*(S)^{odp} \subseteq X$

$S^{odp} \subseteq Y$

$S = s^*(r^*(S))$

$s^*(r^*(S)) = (r \circ s)^*(S) = id_Y^*(S) = S$

$\Rightarrow S^{odp} \subseteq Y$

$r : [0,\infty) \longrightarrow \mathbb{R}^n$

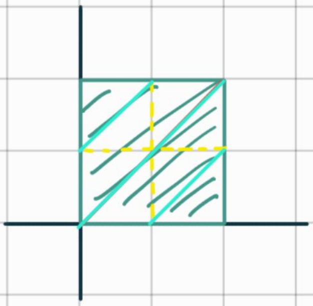
$x \longmapsto (x, 0, \dots, 0)$

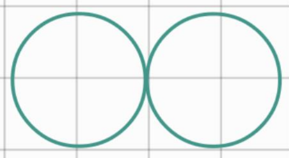
r je prerez

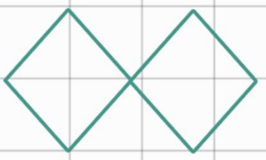
q je retrakcija

$\Rightarrow q$ je kvocientna

e) $[0,1]^2 / \{(t,t);\ t \in [0,1]\}$







$Y := [0,\tfrac{1}{2}] \times [\tfrac{1}{2},1] \cup [\tfrac{1}{2},1] \times [0,\tfrac{1}{2}]$

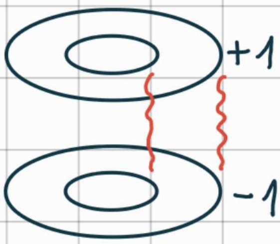
$$(x,y) \longmapsto \begin{cases} (x,y) & ;\ y \geq x+\tfrac{1}{2} \vee y \leq x-\tfrac{1}{2} \\ \left(\tfrac{1}{2} - y + x + \frac{yx - x^2}{1-y+x},\ \tfrac{1}{2} + \frac{yx-x^2}{1-y+x}\right) & ;\ x \geq y \geq x - \tfrac{1}{2} \\ \vdots \\ \vdots \end{cases}$$

f) $(S^n \times [-1,1]) / \{S^n \times \{-1\},\ S^n \times \{1\}\} \approx S^{n+1}$

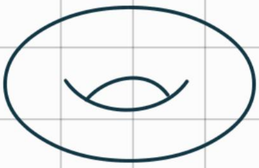
$$f: X \longrightarrow S^{n+1}$$
$$(x,t) \longmapsto (x\sqrt{1-t^2}, t)$$

g) $\{(x,y,z) \in \mathbb{R}^3 ;\ 1 \leq x^2+y^2 \leq 9 \wedge |z| = 1\}/\sim$

$(x,y,z) \sim (u,v,w) :\Longleftrightarrow (x,y,z) = (u,v,w) \vee ((x,y) = (u,v) \wedge x^2+y^2 \in \{1,9\})$



$\approx$



$$f: X \longrightarrow T$$
$$(x,y,z) \longmapsto \left(x, y, z \cdot \sqrt{1^2 - (\sqrt{x^2+y^2} - 2)^2}\right.$$

h) $B^n / S^{n-1} \approx S^n$

$$\begin{array}{ccc} \mathbb{R}^n & & \mathbb{R}^{n+1} \\ \cup\!\!\!\text{I} & & \cup\!\!\!\text{I} \\ f: B^n & \to & S^n \end{array}$$

$$\vec{v} \longmapsto (\lambda \cdot \vec{v}, 2\|\vec{v}\| - 1)$$

$$\left(\begin{array}{l} \lambda^2 \cdot \|\vec{v}\|^2 + (2\|\vec{v}\| - 1)^2 = 1 \\ 4\|\vec{v}\|^4 + \lambda^2 \|\vec{v}\|^2 - 4\|\vec{v}\| = 0 \\ 4\|\vec{v}\| + \lambda^2 \|\vec{v}\| - 4 = 0 \\ \lambda^2 = \frac{4 - 4\|\vec{v}\|}{\|\vec{v}\|} \end{array}\right)$$

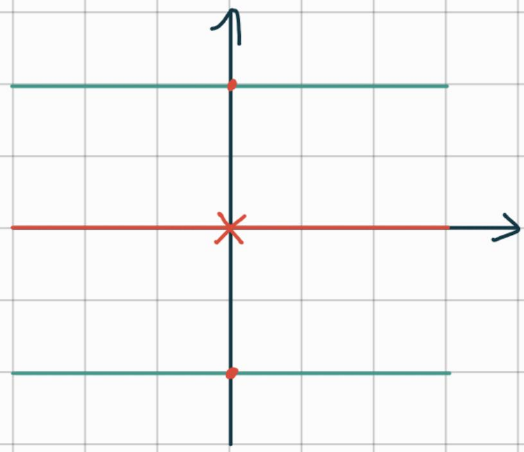
$$\lambda^2 \cdot \|\vec{v}\|^2 + (2\|\vec{v}\| - 1)^2 = 1$$
$$4\|\vec{v}\|^4 + \lambda^2 \|\vec{v}\|^2 - 4\|\vec{v}\| = 0$$
$$4\|\vec{v}\|^4 + \lambda^2 - 4 = 0$$
$$\lambda^2 = 4 - 4\|\vec{v}\|^2$$
$$\lambda = \sqrt{4 - 4\|\vec{v}\|^2}$$
$$\lambda = 2\sqrt{1 - \|\vec{v}\|^2}$$

$$f(\vec{v}) := \left(2\sqrt{1 - \|\vec{v}\|^2} \cdot \vec{v}, 2\|\vec{v}\|^2 - 1\right)$$

1.5) $X := \mathbb{R} \times \{-1, 1\}$

$(x,t) \sim (y,s) \iff (x,t) = (y,s) \vee x = y \neq 0$

Katerim lastnostim T_{0-4} zadošča $X/\sim$?



$X/\sim \in T_1 \iff$ enojci v $X/\sim$ so zaprti $\iff$ ekv. razredi v X so zaprti

Ekvivalenčni razredi so ena točka ali dve.

$\Rightarrow$ So zaprti.

$\Rightarrow$ Je T_1 in T_0.

Trdimo, da $q(0,1)$ in $q(0,-1)$ nimata disjunktnih okolic.

Naj bosta U, V odprti podmnožici v $X/\sim$, da $q(0,1) \in U$ in $q(0,-1) \in V$.

$\underset{\substack{\cup \\ (0,1)}}{q^*(U)}$ in $\underset{\substack{\cup \\ (0,-1)}}{q^*(V)}$ sta odprti v X.

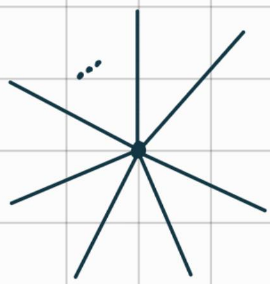
Obstaja $\varepsilon > 0$, tako da $(\varepsilon, 1) \in q^*(U)$ in $(\varepsilon, -1) \in q^*(V)$.

$\Rightarrow q(\varepsilon, 1) = q(\varepsilon, -1) \in U \cap V$

$\Rightarrow$ Ni T_2.

$\Rightarrow$ Ni T_3 in T_4.

1.6) Katerim lastnostim zadošča $(\mathbb{N}\times[0,1])/(\mathbb{N}\times\{1\})$?



T_1 ✓
T_2 ✓

Separabilnost je deljiva. ✓

1-števnost ✗

Lokalna kompaktnost ni deljiva

Naj bo A okolica $[(1,1)]$ v $(\mathbb{N}\times[0,1])/(\mathbb{N}\times\{1\})$.

$([(i,a_i)])_{i\in\mathbb{N}} \subseteq A$

Najdemo lahko odprto okolico $[(1,1)] \subseteq q_*\left(\bigcup_{i\in\mathbb{N}}(\{1\}\times(a_i,1])\right) = V$.

To zaporedje nima stekališča.

Lokalna kompaktnost ✗

1.7) $F: B^n/S^{n-1} \to \mathbb{R}$
$[a] \mapsto \pi\cdot(\|a\|-1)$

Preverite dobro definiranost.

$$\begin{aligned} f: B^n &\to \mathbb{R} \\ a &\mapsto \pi \cdot (\|a\| - 1) \end{aligned}$$

$$\begin{tikzcd} B^n \arrow[d] \arrow[dr, "f"] & \\ B^n/S^{n-1} \arrow[r, "F"'] & \mathbb{R} \end{tikzcd}$$

Preverimo, da je f konstantna na ekvivalenčnih razredih:

$\forall x, y \in S^1: \|x\| = \|y\| = 1 \Rightarrow f(x) = f(y) = \frac{\pi}{2}$

Preverimo, da je F zvezna:

Dovolj je, da je f zvezna, kar pa je, ker je elementarna.

To je geografska širina ..