$d(a,b) = 0 \iff a = b$
$d(a,b) = d(b,a)$
$d(a,c) \leq d(a,b) + d(b,c)$
($d(a,b) \geq 0$ sledi iz ostalih)

1.1) $D((a_1,a_2),(b_1,b_2)) := \begin{cases} |a_1 - b_1| & ; \ b_1 = b_2 \\ |a_1| + |b_1| + |a_2 - b_2| & ; \ \text{sicer} \end{cases}$

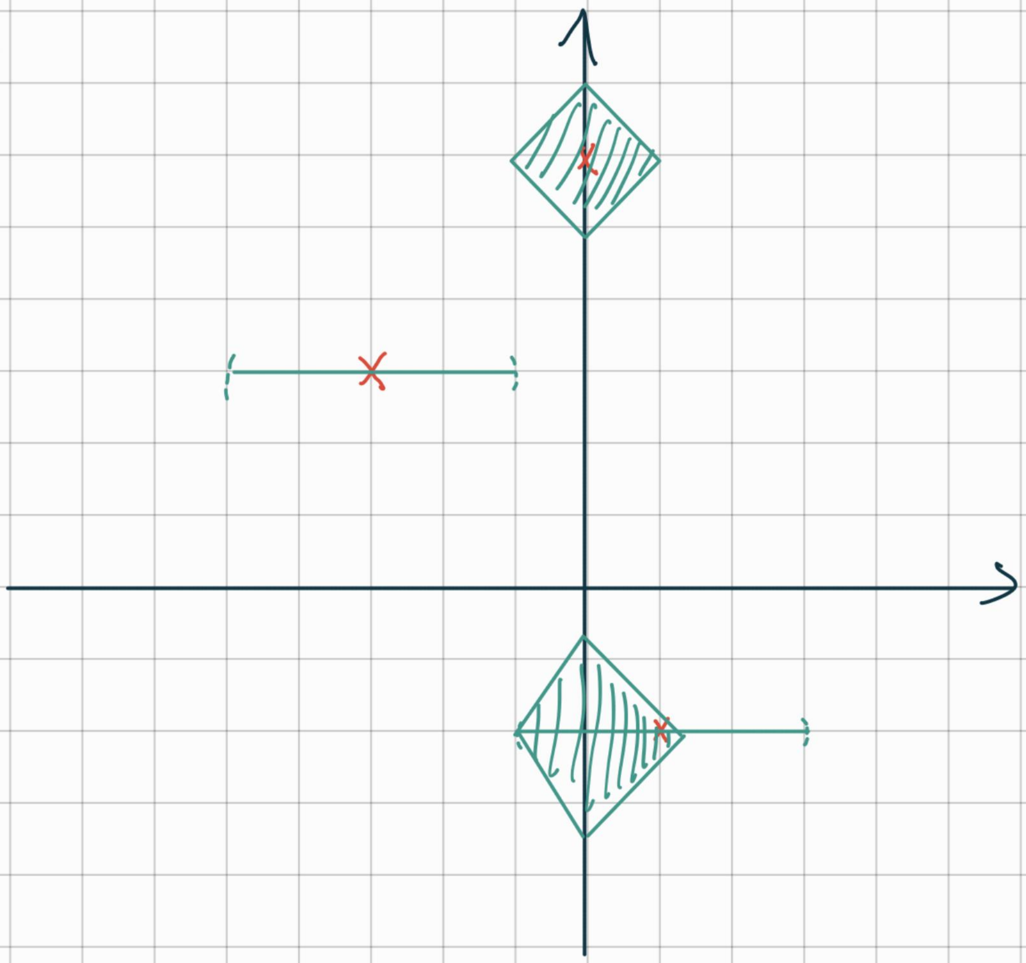
a) $D(a,a) = |a_1 - a_1| = 0$

$D(a,b) = 0$
$a_2 = b_2 \Rightarrow |a_1 - b_1| = 0 \Rightarrow a_1 = b_1$
$a_2 \neq b_2 \Rightarrow |a_1| + |b_1| + |a_2 - b_2| = 0 \Rightarrow a_1 = b_1 = 0 \Rightarrow a_2 = b_2 \Rightarrow \times$

$a_2 = b_2 \Rightarrow |a_1 - b_1| = |b_1 - a_2|$
$a_2 \neq b_2 \Rightarrow |a_1| + |b_1| + |a_2 - b_2| = |b_1| + |a_1| + |b_2 - a_2|$
$\Rightarrow D(a,b) = D(b,a)$

$a_2 = b_2 = c_2 \Rightarrow |a_1 - b_1| + |b_1 - c_1| \geq |a_1 - c_1|$
podobno ostale primere
$\Rightarrow D(a,c) \leq D(a,b) + D(b,c)$

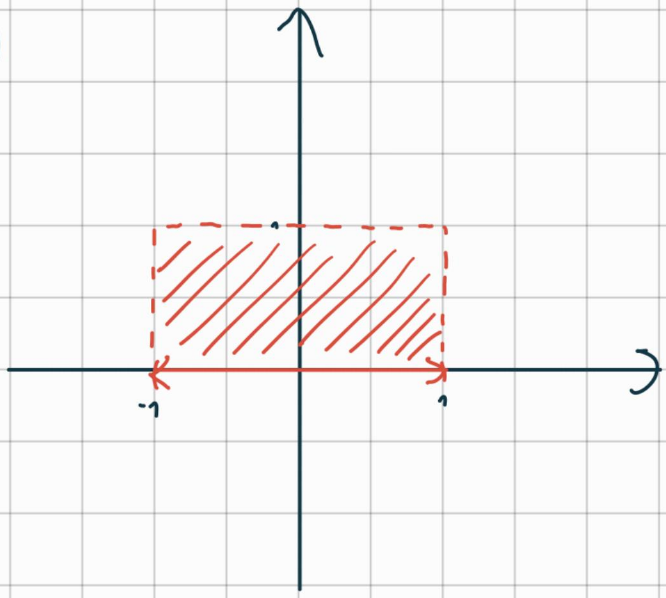
b)



$|a| + |x| + |y-b| < r$

$|x| + |y-b| < r - |a|$

c)



$A = (-1,1) \times [0,1)$

$\mathrm{Int}_D A = A \setminus \{(0,0)\}$

$\mathrm{Cl}_D A = [-1,1] \times [0,1) \cup \{(0,1)\}$

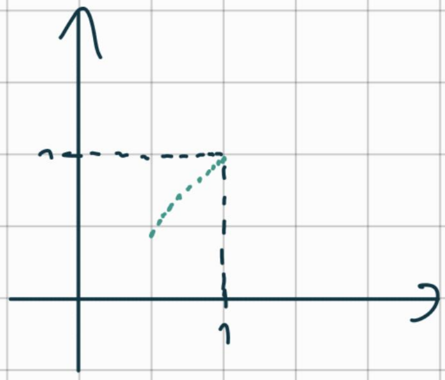
$\partial_D A = \mathrm{Cl} A \setminus \mathrm{Int} A$

$\partial A = \mathrm{Cl} A \setminus \mathrm{Int} A$

$\mathrm{Cl} A = \mathrm{Int} A \cup \partial A = A \cup \partial A$

$\mathrm{Int} A = \mathrm{Cl} A \setminus \partial A$

d)



$\left(\left(\frac{n-1}{n}, \frac{n-1}{n}\right)\right)_{n \in \mathbb{N}}$

V Evklidski metriki konvergira k $(1,1)$.
V metriki D ne konvergira (ni Cauchyjevo).

e) $id : (\mathbb{R}^2, D) \to (\mathbb{R}^2, d)$ je zvezna
$id : (\mathbb{R}^2, d) \to (\mathbb{R}^2, D)$ ni zvezna

1.4) $S \subseteq M$ neprazna
$x \in M$

$$d(x,S) := \inf\{d(x,y) ; y \in S\}$$

a) Dokaži, da je $d(-,S)$ dobro definirana zvezna preslikava.

$\inf\{d(x,y) ; y \in S\}$ je navzdol omejena z 0
$\inf\{d(x,y) ; y \in S\}$ ni prazna, ker S ni prazna

$\Rightarrow$ ima infimum

$f : (M, d_M) \to (N, d_N)$ je C-Lipschitzeva:
$\forall x,y \in M : d_N(f(x), f(y)) \leq C \cdot d_M(x,y)$

$$|d(x,S) - d(y,S)| \leq d(x,y)$$

$$d(x,S) \leq d(x,y) + d(y,S)$$

$$d(x,y) + d(y,S) = d(x,y) + \inf\{d(y,s) ; s \in S\} =$$
$$= \inf\{d(x,y) + d(y,s) ; s \in S\} \geq \inf\{d(x,s) ; s \in S\} =$$
$$= d(x,S)$$

$$-(d(x,S) - d(y,S)) \leq d(x,y)$$

Podobno.

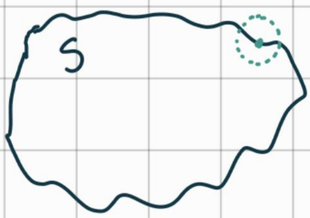
$\Rightarrow d$ je Lipschitzeva

b) $\overline{S} = d(-,S)^*(\{0\})$

($\subseteq$) Naj bo $x \in \overline{S}$.

1) Recimo, da je $x \in S$. Potem $d(x,S) = 0$, torej $x \in d(-,S)^*(\{0\})$.

2) Recimo, da je $x \in \overline{S} \setminus S$. Potem za vsak $r > 0$ velja, da $K(x,r)$ seka S in S^c.



$\Rightarrow \forall r > 0 .\ d(x,S) < r \xrightarrow{r \to 0} d(x,S) = 0$

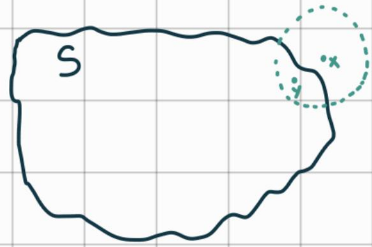
$\Rightarrow x \in d(-,S)^*(\{0\})$

($\supseteq$) Naj bo $x \in d(-,S)^*(\{0\})$.

$\Rightarrow d(x,S) = \inf\{d(x,y) ; y \in S\} = 0$

1) Recimo, da obstaja $y \in S$, da $d(x,y) = 0$. Torej je $x = y \in S$, ker je d metrika na M.

2) Recimo, da tak $y \in S$ ne obstaja. Naj bo $r > 0$. Ker je $\inf\{d(x,y) ; y \in S\} = 0$, obstaja $y \in S$, da je $0 < d(x,y) < r$.



Potem $y \in K(x,r)$, torej $K(x,r)$ seka S. Ker $x \notin S$, $K(x,r)$ seka tudi S^c, torej je $x \in \overline{S}$.

Po definiciji zaprtosti sledi, da je S zaprta v M natanko tedaj, ko je $S = \overline{S} = d(-,S)^*(\{0\})$.

1.2) Prostor $C^\infty([0,1],\mathbb{R})$ opremimo z metriko $d(f,g) := \max_{x\in[0,1]} |f(x)-g(x)|$.

a) Dokažite, da je zaporedje $(x \mapsto \frac{x^n}{n})_n$ konvergentno.

$f_n(x) = \frac{x^n}{n}$
$f(x) = 0$

Očitno.

b) Dokažite, da preslikava $C^\infty([0,1],\mathbb{R}) \to C^\infty([0,1],\mathbb{R})$, $f \mapsto f'$ ni zvezna.

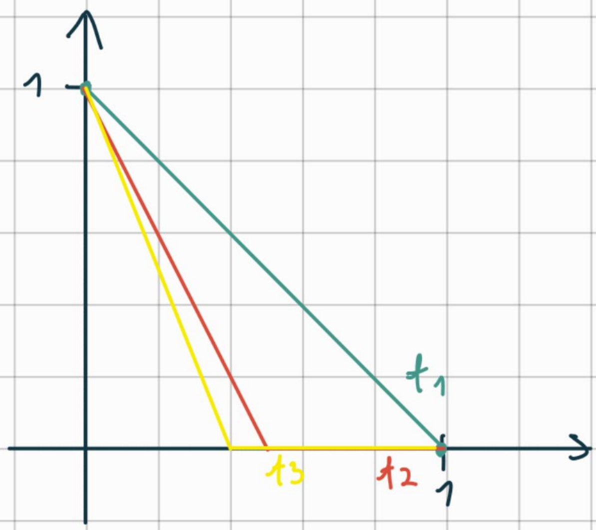
$f_n'(x) = \frac{n \cdot x^{n-1}}{n} = x^{n-1} \xrightarrow[n\to\infty]{} \begin{cases} 1 & ; x=1 \\ 0 & ; \text{sicer} \end{cases}$
$f'(x) = 0$

Torej $(\lim_{n\to\infty} f_n(x))' \neq \lim_{n\to\infty} f_n'(x)$.

1.3) Prostor $C([0,1],\mathbb{R})$ opremimo z metriko $d(f,g) := \int_0^1 |f(x)-g(x)|\,dx$.

a) Obravnavajte konvergenco zaporedij $(x \mapsto \max\{0, 1-n\cdot x\})_n$ in $(x \mapsto \max\{0, n-n^3x\})_n$.

$f_n(x) = \max\{0, 1-n\cdot x\}$

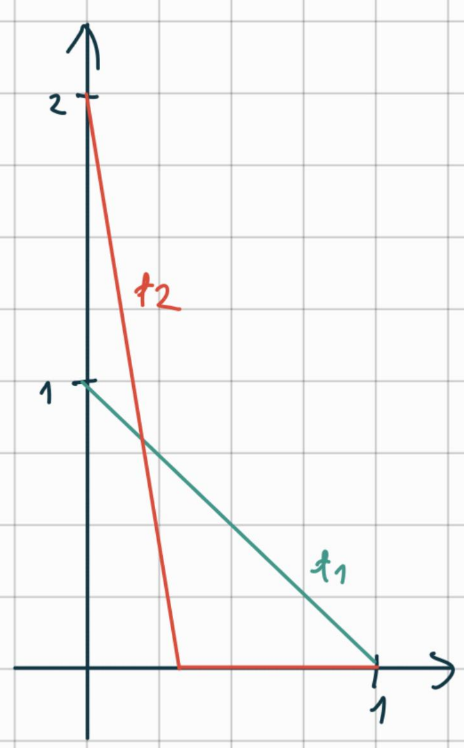


$f(x) = 0$

$$\int_0^1 |f_n(x) - f(x)|\, dx = \int_0^1 |\max\{0, 1-n\cdot x\} - 0|\, dx =$$

$$= \int_0^1 \max\{0, 1-n\cdot x\}\, dx = \int_0^{1/n} (1-n\cdot x)\, dx = \frac{1}{2n} \xrightarrow[n\to\infty]{} 0$$

$g_n(x) = \max\{0, n - n^3\cdot x\}$



? ? ?

b) Dokažite, da zaporedje $(x \mapsto \max\{0, n - n^2 x^3\})_{n\in\mathbb{N}}$ ni Cauchyjevo.

¿?¿

c) Dokažite, da je zaporedje $(x \mapsto \min\{\max\{0, nx - \frac{n}{2}\}, 1\})_{n \in \mathbb{N}}$ Cauchyjevo, ne pa konvergentno.

$$f(x) = \begin{cases} 0 ; & 0 \leq x < \frac{1}{2} \\ 1 ; & \frac{1}{2} \leq x \leq 1 \end{cases} \quad \notin C([0,1], \mathbb{R})$$

To zaporedje bi bilo konvergentno proti f, ki pa ni zvezna, torej ni konvergentno. Je pa Cauchyjevo ...

1.5) Naj bo M metrični prostor z metriko d in naj $\mathcal{M}$ označuje množico vseh nepraznih kompaktnih podmnožic prostora M. Definirajmo preslikavo $hd : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ s predpisom:

$$hd(K, L) := \max\{\sup_{x \in K} d(x, L), \sup_{y \in L} d(y, K)\}.$$

a) Dokažite, da je hd metrika na $\mathcal{M}$ (Hausdorffova metrika).

$\underline{hd(K,L) = 0 \Rightarrow K = L}$

$$hd(K,L) = 0$$

$$\Rightarrow \sup_{x \in K} d(x, L) = 0 \text{ in } \sup_{y \in L} d(y, K) = 0$$

$$\Rightarrow \forall x \in K. \, d(x, L) = 0 \text{ in } \forall y \in L. \, d(y, K) = 0$$

$$\Rightarrow \forall x \in K. \, x \in \overline{L} \text{ in } \forall y \in L. \, y \in \overline{K}$$

$$\Rightarrow \overline{K} = \overline{L}$$

K, L kompaktni

$\Rightarrow K = L$

$K = L \Rightarrow hd(K, L) = 0$

$\forall x \in L.\ d(x, K = L) = 0 \Rightarrow \sup \ldots = 0$
$\forall y \in K.\ d(y, L = K) = 0 \Rightarrow \sup \ldots = 0$

$\Rightarrow d(K, L) = 0$

$hd(K, L) = hd(L, K)$

Očitno.

$hd(A, C) \leq hd(A, B) + hd(B, C)$

???

b) Pokažite, da predpis $x \mapsto \{x\}$ določa izometrično vložitev $(M, d) \rightarrow (\mathcal{M}, hd)$.

$x, y \in M$:

$hd(\{x\}, \{y\}) =$

$= \max\{\sup_{x \in \{x\}} d(x, \{y\}), \sup_{y \in \{y\}} d(y, \{x\})) =$

$= \max\{d(x, \{y\}), d(y, \{x\})\} =$

$= \max\{\inf_{y \in \{y\}} d(x, y), \inf_{x \in \{x\}} d(y, x)\} =$

$$= \max\{d(x,y), d(y,x)\} =$$

$$= d(x,y)$$