REKURZIVNE ENAČBE

$a_{n+3} - 2a_{n+2} - 4a_{n+1} + 8a_n = 0$
$a_0 = 0,\ a_1 = 1,\ a_2 = 2$

$x^3 - 2x^2 - 4x + 8 = 0$
$x^2(x-2) - 4(x-2) = 0$
$(x-2)^2(x+2) = 0$
$x_{1,2} = 2,\ x_3 = -2$

$a_n = (A + B\cdot n)\cdot 2^n + C\cdot(-2)^n$

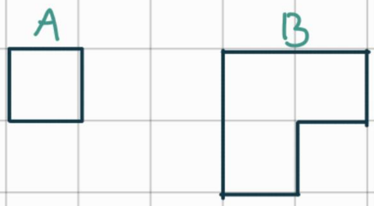
$0 = A + C$
$1 = 2A + 2B - 2C$
$2 = 4A + 8B + 4C$

$\Rightarrow A = \frac{1}{8}$
$B = \frac{1}{4}$
$C = -\frac{1}{8}$

$a_n = \left(\frac{1}{8} + \frac{1}{4}n\right)\cdot 2^n - \frac{1}{8}\cdot(-2)^n$

šahovnica $2\times n$

ploščice:



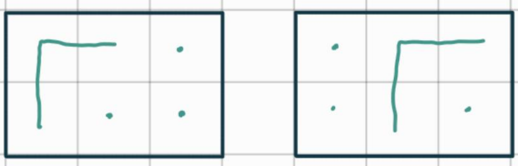
\+ rotacije

a_n ... število načinov za pokritje šahovnice $2 \times n$ s temi ploščicami

$a_1 = 1$

$a_2 =$ sami A + en A in en B · rotacije $= 1 + 4 = 5$

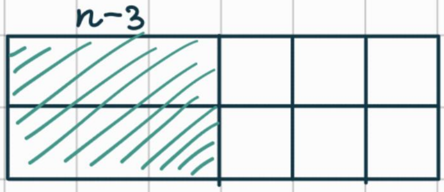
2B: 2

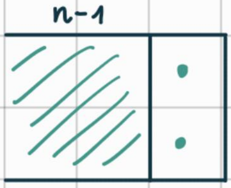


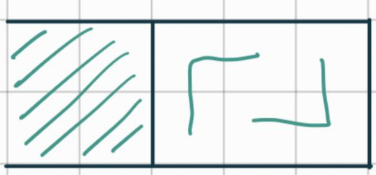
1B: 4+4

0B: 1

$a_3 = 11$







same A-je štejemo že prej

$a_n = 1 \cdot a_{n-1} + 4 \cdot a_{n-2} + 2 \cdot a_{n-3}$

$x^3 - x^2 - 4x - 2 = 0$

$(x+1)(x^2 - 2x - 2) = 0$

$x_1 = -1$
$x_{2,3} = 1 \pm \sqrt{3}$

$a_n = (-1)^n \cdot A + (1+\sqrt{3})^n \cdot B + (1-\sqrt{3})^n \cdot C$

$a_0 = A + B + C = 1$
$a_1 = -A + (1+\sqrt{3})B + (1-\sqrt{3})C = 1$
$a_2 = A + (1+\sqrt{3})^2 B + (1-\sqrt{3})^2 C = 5$

$A = 1$
$B = \frac{1}{\sqrt{3}}$
$C = -\frac{1}{\sqrt{3}}$

---

a_n = stevilo nizov dolžine n iz znakov $0, 1$, ki ne vsebujejo podniza 00

$a_0 = 1$ (prazen niz)
$a_1 = 2$ $(0, 1)$
$a_2 = 3$ $(01, 10, 11)$
$a_3 = 5$

i) $\underline{\quad n-1 \quad}\ 1$

ii) $\underline{\quad n-2 \quad}\ 10$

$a_n = a_{n-1} + a_{n-2}$

---

a_n = stevilo nizov iz znakov a, b, c, ki ne vsebujejo podniza ab

$a_0 = 1$

$a_1 = 3$

$a_2 = 3^2 - 1 = 8$

$a_3 = 3^3 - 6 = 21$

i) $\underbrace{\_\_\_\_\_\_\_\_\_\_\_\ a}_{n}$ $\sim a_{n-1}$

ii) $\underbrace{\_\_\_\_\_\_\_\_\_\_\_\ c}_{n}$ $\sim a_{n-1}$

iii) $\_\_\_\_\_\_\_\_\_\_\_\ b$

a) $\_\_\_\_\_\_\_\_\_\_\_\ c\ b$ $\sim a_{n-2}$

b) $\_\_\_\_\_\_\_\_\_\_\_\ b\ b$

I) $\_\_\_\_\_\_\_\_\_\_\_\ c\ b\ b$ $\sim a_{n-3}$

II) ...

*) $\_\_\_\_\ b\ b\ b\ b$

$a_n = 2a_{n-1} + a_{n-2} + a_{n-3} + \ldots + 2$

$a_{n-1} = 2a_{n-2} + a_{n-3} + a_{n-4} + \ldots + 2$

odštejemo:

$a_n - a_{n-1} = 2a_{n-1} - a_{n-2} + 0$

$\Rightarrow a_n = 3a_{n-1} - a_{n-2}$

b_n = število nizov dolžine n iz a, b, c, ki ne vsebujejo ab in se ne končajo z a

$$a_n = 2a_{n-1} + b_{n-1}$$

$$b_n : \underset{b_{n-1}}{\rule{3cm}{0.4pt}}\, b \qquad\qquad \underset{a_{n-1}}{\rule{3cm}{0.4pt}}\, c$$

$$b_n = b_{n-1} + a_{n-1}$$

$$\Rightarrow a_n - b_n = 2a_{n-1} - a_{n-1}$$

$$b_n = a_n - a_{n-1}$$

$$\Rightarrow b_{n-1} = a_{n-1} - a_{n-2}$$

$$x^2 = 3x - 1$$
$$x^2 - 3x + 1 = 0$$

$$x_{1/2} = \frac{3 \pm \sqrt{5}}{2}$$

$$a_n = A \cdot \left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)^n + B \cdot \left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right)^n$$

$$a_0 = A + B = 1 \Rightarrow B = 1 - A$$

$$a_1 = A\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right) + B\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right) = 3$$

$$a_n = F_{2n+1}$$

$$F_n = F_{n-1} + F_{n-2}$$
$$a_n = 3a_{n-1} - a_{n-2}$$

I.P.: $a_i = F_{2i+1}$ za $i = 0, \dots, n$

$n=0$:

$$a_0 = 1$$
$$F_1 = 1$$

$n \to n+1$:

$$a_{n+1} = 3a_n - a_{n-1} \overset{\text{I.P.}}{=} 3F_{2n+1} - F_{2n-1} \overset{F_{2n+1}=F_{2n}+F_{2n-1}}{=}$$

$$= 2\cdot F_{2n+1} + F_{2n} + F_{2n-1} - F_{2n-1} = F_{2n+1} + F_{2n+1} + F_{2n} =$$

$$= F_{2n+1} + F_{2n+2} = F_{2n+3}$$

KOMPLEKSNE NIČLE

$$\lambda = x + iy = |\lambda|(\cos\varphi + i\sin\varphi)$$
$$\overline{\lambda} = x - iy = |\lambda|(\cos\varphi - i\sin\varphi)$$

$$a_n = A\lambda^n + B\overline{\lambda}^n =$$

$$= |\lambda|^n\left(A(\cos n\varphi + i\sin n\varphi) + B(\cos n\varphi - i\sin n\varphi)\right) =$$

$$= |\lambda|^n\left((A+B)\cos n\varphi + (Ai - Bi)\sin n\varphi\right) =$$

$$= |\lambda|^n \cdot \left(A'\cos n\varphi + B'\sin n\varphi\right)$$

$$D_n = \begin{vmatrix} b & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ b & b & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & b & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b & b & b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & b & b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b & b \end{vmatrix}, \quad b>0$$

$$D_1 = |b| = b$$

$$D_2 = \begin{vmatrix} b & b \\ b & b \end{vmatrix} = b^2 - b^2 = 0$$

$$D_3 = \begin{vmatrix} b & b & 0 \\ b & b & b \\ 0 & b & b \end{vmatrix} = b \cdot \begin{vmatrix} b & b \\ b & b \end{vmatrix} - b \begin{vmatrix} b & 0 \\ b & b \end{vmatrix} = 0 - b^3 = -b^3$$

$$D_n = b \cdot D_{n-1} - b \cdot \begin{vmatrix} b & 0 & 0 & \cdots & & 0 \\ b & b & b & 0 & \cdots & 0 \\ 0 & b & b & b & 0 \cdots & 0 \\ & & & & & \vdots \\ & & & & & 0 \\ & & & 0 & b & b \end{vmatrix} =$$

$$= b \cdot D_{n-1} - b \cdot b \cdot D_{n-2}$$

$$\Rightarrow D_n = b D_{n-1} - b^2 D_{n-2}$$

$$x^2 - bx + b^2 = 0$$

$$x_{1,2} = \frac{b \pm \sqrt{b^2 - 4b^2}}{2} = \frac{b \pm i b\sqrt{3}}{2}$$

$$|x| = \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{b\sqrt{3}}{2}\right)^2} = \frac{1}{2}\sqrt{b^2 + 3b^2} = b$$

$$\varphi = 60^\circ$$

$$D_n = b^n \left(A \cos\left(n\frac{\pi}{3}\right) + B \sin\left(n\frac{\pi}{3}\right)\right)$$

$$D_1 = b\left(A\cdot\frac{1}{2} + B\cdot\frac{\sqrt{3}}{2}\right) = b$$
$$D_2 = b^2\left(A\cdot\left(-\frac{1}{2}\right) + B\cdot\frac{\sqrt{3}}{2}\right) = 0$$

$$\left.\begin{array}{l} A + B\sqrt{3} = 2 \\ -A + B\sqrt{3} = 0 \end{array}\right\} +$$

$$2\sqrt{3}B = 2 \Rightarrow B = \frac{\sqrt{3}}{3}$$
$$2A = 2 \Rightarrow A = 1$$

$$D_n = b^n\left(\cos\left(n\frac{\pi}{3}\right) + \frac{\sqrt{3}}{3}\sin\left(n\frac{\pi}{3}\right)\right)$$

---

a_n = stevilo nizov dolzine n iz znakov $0,1,2,3$, pri katerih se 3 nikoli ne pojavi za 0

331102 je ok
201133 ni ok

$$a_0 = 1$$
$$a_1 = 4$$
$$a_2 = 4^2 - 1 = 15$$

$\underbrace{\qquad\qquad\qquad}_{n-1}3 \sim 3^{n-1}$ (pred 3 so lahko vsi nizi iz 1,2,3)

$\underbrace{\qquad\qquad\qquad}_{n-1}0/1/2 \sim 3\cdot a_{n-1}$ (vsi krajši ustrezni)

$$\Rightarrow a_n = 3^{n-1} + 3a_{n-1}$$

$$\Rightarrow a_n - 3a_{n-1} = 3^{n-1}$$

$$a_n = z_n + b_n$$

homogena:

$$z_n - 3z_{n-1} = 0$$
$$x - 3 = 0$$
$$x = 3$$

$$\Rightarrow z_n = A \cdot 3^n$$

nastavek za desno stran:

$$f(n) = p(n) \cdot \alpha^n$$

$$b_n = q(n) \cdot n^a \cdot \alpha^n$$

$$\deg(q) \leq \deg(p)$$
a ... kratnost α v karakterističnem polinomu homogene enačbe

posebna rešitev nehomogene:

$$b_n = B \cdot 3^n \cdot n$$

vstavimo v enačbo:

$$b_n = 3^{n-1} + 3b_{n-1}$$
$$B \cdot 3^n \cdot n = 3^{n-1} + 3 \cdot B \cdot 3^{n-1} \cdot (n-1)$$

$$n: B \cdot 3^n = B \cdot 3^n$$
$$1: \quad 0 = 3^{n-1} - 3^n \cdot B$$

$$\Rightarrow B = \frac{1}{3}$$

$$a_n = A \cdot 3^n + \frac{1}{3} \cdot 3^n \cdot n = A \cdot 3^n + 3^{n-1} \cdot n$$

$a_0 = 1 = A \cdot 3^0 + 0 = A$

$\Rightarrow a_n = 3^n + 3^{n-1} \cdot n$

RODOVNE FUNKCIJE

Zaporedju (a_n) ustreza formalna potenčna vrsta
$A(x) = \sum_{n=0}^{\infty} a_n x^n$.

$A(x) = \sum_{n=0}^{\infty} a_n x^n$
$B(x) = \sum_{n=0}^{\infty} b_n x^n$

$C(x) = A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n$
$c_n = \sum_{i=0}^{n} a_i b_{n-i}$

$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$

---

a) $1, 0, 1, 0, 1, 0, \ldots$

$A(x) = 1 + x^2 + x^4 + \ldots = \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}$

b) $0, 1, 0, 1, 0, 1, \ldots$

$B(x) = x + x^3 + x^5 + \ldots = x \sum_{n=0}^{\infty} x^{2n} = \frac{x}{1-x^2}$

c) $1, 2, 3, 3, 3, \ldots$

$C(x) = 1 + 2x + 3x^2 + 3x^4 + \ldots = 1 + 2x + 3x^2 \cdot \frac{1}{1-x}$

d) $1, n, \binom{n}{2}, \binom{n}{3}, \dots, \binom{n}{n}, 0, 0, \dots$

$$D(x) = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n = (1+x)^n$$

(a_n), $A(x) = \sum a_n x^n$

a) (p_n): $p_n = 2a_n$

b) (q_n): $q_n = a_n + 2$

c) (r_n): $r_n = a_{n-2}$

a) $P(x) = \sum_{n=0}^{\infty} 2a_n x^n = 2\sum_{n=0}^{\infty} a_n x^n = 2A(x)$

b) $Q(x) = \sum_{n=0}^{\infty} (a_n+2)x^n = \sum_{n=0}^{\infty} a_n x^n + 2\sum_{n=0}^{\infty} x^n = A(x) + \frac{2}{1-x}$

c) $R(x) = \sum_{n=2} a_{n-2} x^n = x^2 \sum_{n=0}^{\infty} a_n x^n = x^2 A(x)$

Zapiši rodovno funkcijo za število razčlenitev števila n na sumande velikosti:

a) 3
b) 5
c) 3 ali 5

a) $a_0 = 1$
$a_1 = 0$
$a_2 = 0$
$a_3 = 1$
$a_4 = 0$
$a_5 = 0$
$a_6 = 1$

$$A(x) = 1 + x^3 + x^6 + \dots = \sum_{n=0}^{\infty} \frac{1}{1-x^3}$$

b) $B(x) = \sum_{n=0}^{\infty} \frac{1}{1-x^5}$

k zapišemo kot vsoto trojk
n-k zapišemo kot vsoto petk

c) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ ... člen v produktu

$$C(x) = A(x) \cdot B(x) = \frac{1}{1-x^3} \cdot \frac{1}{1-x^5}$$

Na koliko načinov lahko izplačamo znesek 13 EUR s kovanci in bankovci po 1, 2, 5, 10 EUR?

a_n ... število načinov za izplačilo n EUR

$A_i(x)$ ... rodovna funkcija za izplačilo n EUR z bankovci za i EUR

$A_1(x) = 1 + x + x^2 + \dots = \frac{1}{1-x}$
$A_2(x) = 1 + x^2 + \dots = \frac{1}{1-x^2}$
$A_5(x) = \frac{1}{1+x^5}$
$A_{10}(x) = \frac{1}{1+x^{10}}$

$$A(x) = A_1(x)\, A_2(x)\, A_5(x)\, A_{10}(x)$$

Komentar: Razmislek kot v prejšnji nalogi.

$$A_1(x)\, A_2(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} = (1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots) =$$

$$= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + $$
$$+ 5x^8 + 5x^9 + 6x^{10} + 6x^{11} + 7x^{12} + 7x^{13} + \dots$$

$$A_5(x)\, A_{10}(x) = (1 + x^5 + x^{10} + \dots)(1 + x^{10} + \dots) =$$

$$= 1 + x^5 + 2x^{10} + \dots$$

$$a_{13} = 7 + 5 + 4 = 16$$

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24 enakih jabolk
4 otroci
vsak vsaj 3 in največ 8 jabolk

a_n ... število načinov za razdelitev n jabolk med 4 otroke

$A_i(x)$ ... rodovna funkcija za otroka i

$$A_i(x) = 0 + 0x + 0x^2 + 1x^3 + 1x^4 + \dots + 1x^8 + 0 + \dots =$$

$$= x^3(1 + x + x^2 + x^3 + x^4 + x^5)$$

$$A(x) = (A_i(x))^4 = x^{12}(1 + x + x^2 + x^3 + x^4 + x^5)^4 =$$

$$= x^{12} \cdot \left(\frac{1-x^6}{1-x}\right)^4 = x^{12}(1-x^6)^4(1-x)^{-4} =$$

$$= x^{12} \cdot \sum_{i=0}^{4} \binom{4}{i}(x^6)^i \cdot \sum_{j=0}^{\infty} \binom{-4}{i}(-x)^j =$$

$$= x^{12} \cdot \sum_{i=0}^{4} \binom{4}{i}(x^6)^i \cdot \sum_{k=0}^{\infty} \binom{k+4-1}{4-1}(-x)^k$$

Koeficient pri x^{24}:

$i=0,\ k=12:\quad 1\cdot\binom{15}{3}$
$i=1,\ k=6:\quad -4\cdot\binom{9}{3}$
$i=2,\ k=0:\quad \binom{4}{2}\cdot 1$

Seštejemo:

$a_{24} = 125$

Koliko je celoštevilskih rešitev enačbe $x_1+x_2+x_3+x_4=20$?

a) $2 \le x_i \le 7$
b) $x_i \ge 0$, x_2, x_3 sodi

a) a_n ... število rešitev enačbe $x_1+x_2+x_3+x_4=n$,
kjer je $2 \le x_i \le 7$

$A_i(x)$ ... rodovna funkcija za spremenljivko x_i

$$A_i(x) = x^2+x^3+\dots+x^7$$

$$A(x) = A_1(x)A_2(x)A_3(x)A_4(x)$$

Poiščemo a_{20} ...

b) b_n ... število rešitev enačbe $x_1+x_2+x_3+x_4=n$,
kjer je $x_i \ge 0$ in x_2, x_3 soda

$$B_1(x) = 1+x+x^2+x^3+\dots = \frac{1}{1-x}$$
$$B_4(x) = \frac{1}{1-x}$$

$$B_2(x) = 1+x^2+x^4+x^6+\dots = \frac{1}{1-x^2}$$
$$B_3(x) = \frac{1}{1-x^2}$$

$$B(x) = (B_1(x))^2 (B_2(x))^2 = \left(\frac{1}{1-x}\right)^2 \left(\frac{1}{1-x^2}\right)^2$$

Iščemo b_{20} ...

S pomočjo rodovnih funkcij pokaži:

a) Število razčlenitev n, pri katerih so vsi sumandi sodi, je enako številu razčlenitev n, pri kateri se vsak sumand pojavi sodo mnogokrat.

a_n ... število razčlenitev n, vsi sumandi sodi
b_n ... število razčlenitev n, vsak sumand sodo mnogokrat

$$A(x) = \frac{1}{1-x^2} \cdot \frac{1}{1-x^4} \cdot \frac{1}{1-x^6} \cdot \frac{1}{1-x^8} \cdot \ldots$$

$$B_1(x) = 1 + x^2 + x^4 + x^6 + \ldots = \frac{1}{1-x^2}$$
$$B_2(x) = 1 + x^4 + x^8 + \ldots = \frac{1}{1-x^4}$$
$$B_i(x) = 1 + x^{2i} + x^{4i} + \ldots = \frac{1}{1-x^{2i}}$$

$$B(x) = \frac{1}{1-x^2} \cdot \frac{1}{1-x^4} \cdot \frac{1}{1-x^6} \cdot \frac{1}{1-x^8} \cdot \ldots$$

b) Število razčlenitev n, pri katerih se noben sumand ne pojavi več kot dvakrat, je enako številu razčlenitev n, pri katerih noben sumand ni deljiv s 3.

a_n ... število razčlenitev n, noben sumand več kot $2\times$
b_n ... število razčlenitev n, noben sumand deljiv s 3

$$A_1(x) = 1 + x + x^2$$
$$A_2(x) = 1 + x^2 + x^4$$
$$A_3(x) = 1 + x^3 + x^6$$
$$\vdots$$

$$A(x) = (1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)\cdot\ldots$$

$$B(x) = \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^4}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\cdot\frac{1}{1-x^8}\cdot\frac{1}{1-x^{10}}\cdot\ldots$$

$$1+a+a^2+\ldots+a^k = \frac{1-a^{k+1}}{1-a}$$

$$A(x) = \frac{\cancel{1-x^3}}{1-x}\cdot\frac{1-(x^2)^3}{1-x^2}\cdot\frac{1-(x^3)^3}{\cancel{1-x^3}}\cdot\frac{1-(x^4)^3}{1-x^4}\cdot\ldots$$

Ustrezni se pokrajšamo in dobimo isto...

CATALANOVA STEVILA

$C_0 = C_1 = 1$

$C_2 = 2$

$C_3 = 5$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0$$

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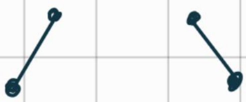
a_n = # dvojiških dreves s korenom na n vozliščih

$a_1 = 1$

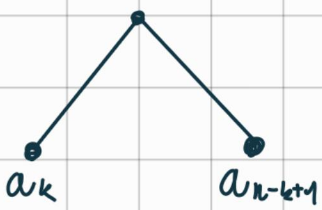
•

$a_2 = 2$



$a_3 = 5$

a_n:



$$a_n = \sum_{k=0}^{n-1} a_k \, a_{n-k-1}$$

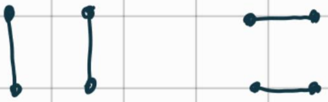
$$\Rightarrow a_n = C_n = \frac{1}{n+1}\binom{2n}{n}$$

---

a_n = # načinov, na katere se lahko rokuje $2n$ ljudi, ki sedijo okoli okrogle mize, tako, da se vsak rokuje z enim drugim, da se noben par ne križa

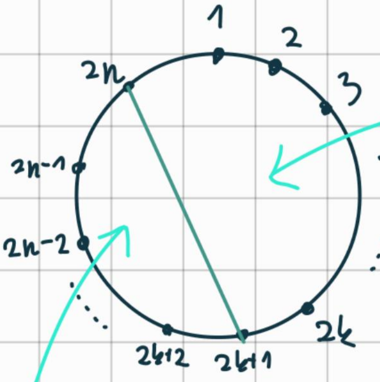
$a_1 = 1$:

$a_2 = 2$



$a_3 = 2+3 = 5$

a_n :



2k ljudi
a_k načinov

$2n-1-(2k+2)+1 = 2n-2k-2$ ljudi
a_{n-k+1} načinov

$$a_n = \sum_{k=0}^{n-1} a_k a_{n-k+1} = C_n$$

---

a_n = # poti od $(0,0)$ do $(2n,0)$, ki nikoli ne zaidejo pod x-os, če uporabljamo korake $(1,1)$ in $(1,-1)$

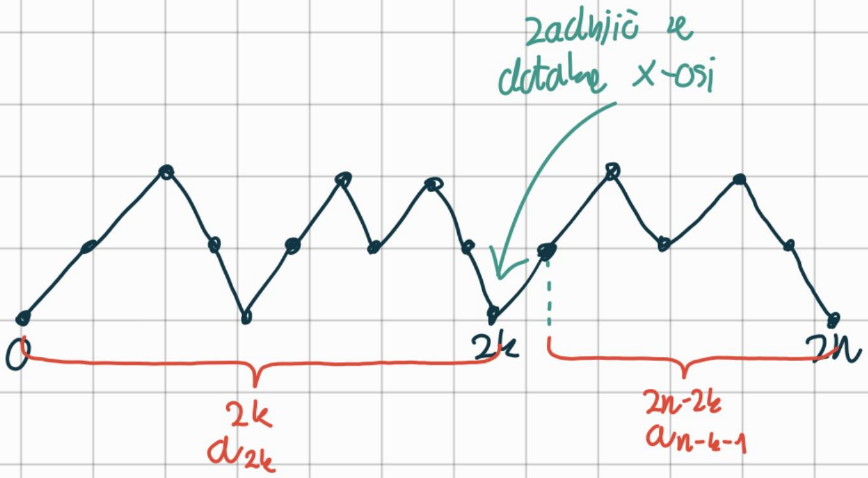
$a_0 = 1$

$a_1 = 1$

$a_2 = 2$

$a_3 = 5$

a_n:



$$a_n = \sum_{k=0}^{\infty} a_k a_{n-k-1} = c_n$$

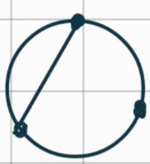
Motzkinovo število M_n je število načinov, na katere lahko narišemo tetive na krogu med n danimi točkami tako, da nobeni dve tetivi nimata skupne točke.

$M_0 = 1$

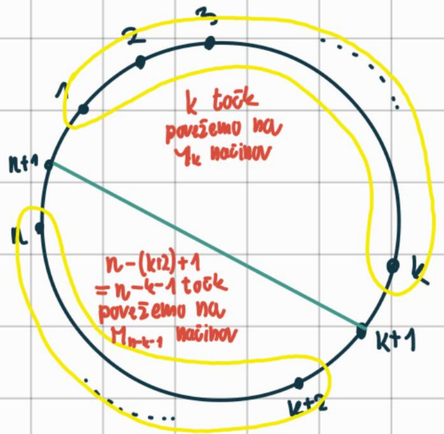
$M_1 = 1$

$M_2 =$ nič ali ena $= 1 + 1 = 2$

$M_3 =$ nič ali ena $= 1 + 3 = 4$



$M_4 =$ nič ali ena ali dve $= 1 + \binom{4}{2} + 2 = 9$



$$k \in \{0, 1, \dots, n-1\}$$

$$M_{n+1} = \underbrace{\sum_{k=0}^{n-1} M_k M_{n-k-1}}_{n+1 \text{ krajišče tetive}} + \underbrace{M_n}_{n+1 \text{ ni krajišče tetive}} \quad , \quad n \geq 0$$

$$\sum_{n=0}^{\infty} M_{n+1} x^{n+1} = \sum_{n=0}^{\infty} M_n x^{n+1} + \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} M_k M_{n-k-1} x^{n+1} =$$

$$= M(x) - M_0 = x M(x) + \sum_{n=0}^{\infty} \sum_{k=0}^{n} M_k M_{n-k} x^{n+2} =$$

$$= x M(x) + x^2 M(x)^2$$

$$x^2 M(x)^2 + (x-1) M(x) + 1 = 0$$

$$M(x) = \frac{1 - x - \sqrt{(x-1)^2 - 4x^2}}{2x^2}$$

mjav

ODVOD POTENČNE VRSTE

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$A'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$a_n = 1$$

$$A(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$A'(x) = \sum_{n=0}^{\infty} (n+1) x^n = B(x) + A(x) = \frac{1}{(1-x)^2}$$

$b_n = n$

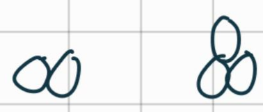
$$B(x) = \sum_{n=0}^{\infty} n x^n = A'(x) - A(x) = \frac{1}{(1-x)^2} - \frac{1}{1-x} = \frac{x}{(1-x)^2}$$

---

a_n = # načinov, na katere enake kovance razporedimo v vrste, da je vsaka vrsta neprekinjena in se vsak kovanec v višji vrsti dotika natanko dveh kovancev v spodnji vrsti, in v najspodnejši vrsti je natanko n kovancev

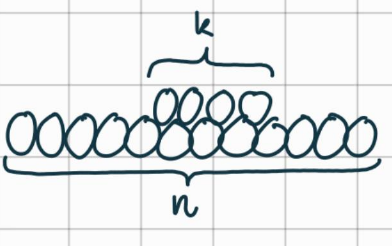
$a_1 = 1$

$a_2 = 2$



$a_3 = 5$

$a_4 = 13$



Imamo n-k pozicij gornjega kupčka.

$$a_n = \sum_{k=1}^{n-1} (n-k) a_k + 1$$

$$A(x) = \sum_{n \geq 0} a_n x^n$$

$$a_0 := 0$$

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \sum_{k=\cancel{1}\,0}^{\cancel{n-1}\,n} (n-k) a_k x^n + \sum_{n=1}^{\infty} x^n$$

lahko vzamemo od $n=0$
ker $(n-0) a_0 x^0 = 0$

$$A(x) - \underset{0}{\underbrace{a_0}} = A(x) \cdot B(x) + \frac{x}{1-x}$$

$$A(x) = A(x) \cdot \frac{x}{(1-x)^2} + \frac{x}{1-x}$$

$$A(x) \left(1 - \frac{x}{(1-x)^2}\right) = \frac{x}{1-x}$$

$$A(x) = \frac{x(1-x)}{1-3x+x^2}$$