KOMPLEKSNA ŠTEVILA

Kompleksno število je urejen par (x,y), ki ga po navadi pišemo $z=x+yi$, kjer je i imaginarna enota, za katero velja $i^2=-1$. Realni del števila z je $Re(z)=x$, imaginarni del pa $Im(z)=y$.

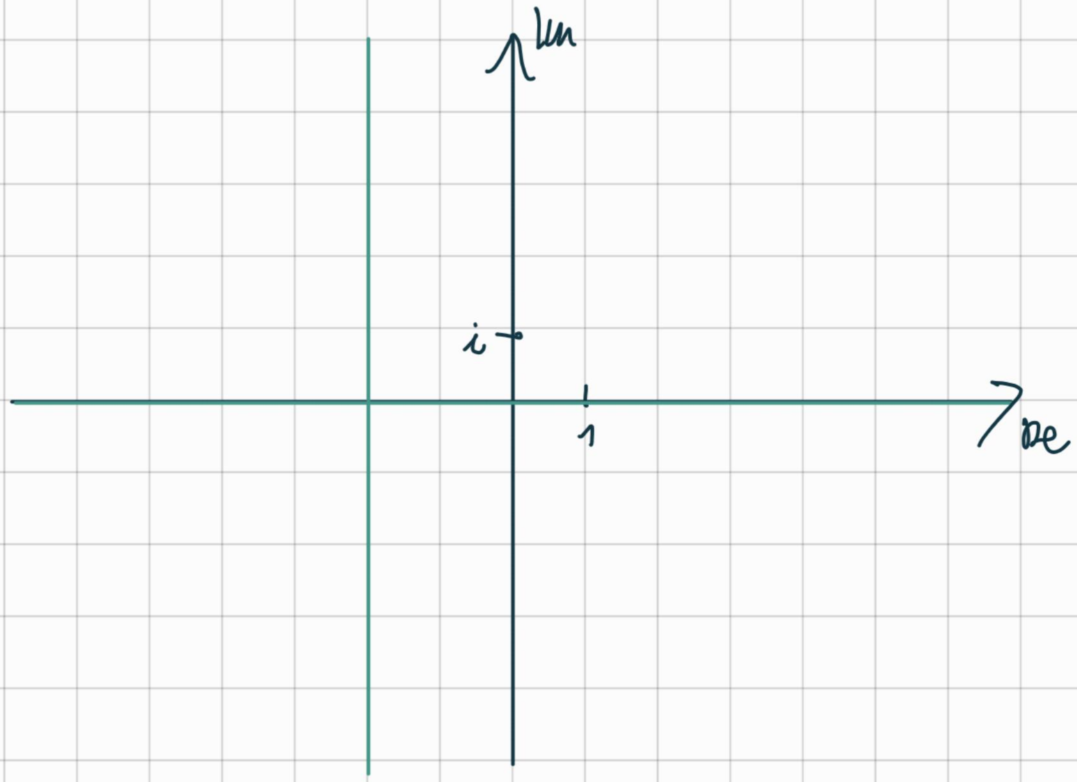
Konjugirano število k številu z je $\bar{z}=x-yi$.
Velja $Re(z)=\frac{z+\bar{z}}{2}$ in $Im(z)=\frac{z-\bar{z}}{2i}=i\cdot\frac{\bar{z}-z}{2}$.

$$(1-2i)^4=\binom{4}{0}+\binom{4}{1}\cdot(-2i)+\binom{4}{2}\cdot(-2i)^2+\binom{4}{3}\cdot(-2i)^3+\binom{4}{4}(-2i)^4$$
$$=1-8i+6(-4)+4\cdot 8i+16$$
$$=-7+24i$$

$$z^2+4z-1\in\mathbb{R}\quad,\quad z\in\mathbb{C}$$

$$(x+yi)^2+4(x+yi)-1=$$
$$=x^2+2xyi-y^2+4x+4yi-1=$$
$$=(x^2-y^2+4x-1)+i(2xy+4y)$$

$$2xy+4y=0$$
$$y(x+2)=0$$
$$y=0 \text{ ali } x=-2$$



Enačbo bi lahko rešili tudi drugače:

$$z^2+4z-1 \in \mathbb{R} \iff z^2+4z-1 = \overline{z^2+4z-1}$$

$$z^2+4z-1 = \overline{z^2+4z-1}$$
$$z^2+4z-1 = \bar{z}^2+4\bar{z}-1$$
$$z^2-\bar{z}^2+4z-4\bar{z}=0$$
$$(z-\bar{z})(z+\bar{z})+4(z-\bar{z})=0$$
$$(z-\bar{z})(z+\bar{z}+4)=0$$

$z=\bar{z}$	ali	$z+\bar{z}+4=0$
$\mathrm{Im}(z)=0$		$2\,\mathrm{Re}(z)+4=0$
$z \in \mathbb{R}$		$\mathrm{Re}(z)=-2$

Absolutna vrednost števila z je $|z| = \sqrt{x^2+y^2} = \sqrt{z\cdot\bar{z}}$.
Število $|z|$ je oddaljenost števila z od koordinatnega izhodišča, izraz $|z-w|$ pa razdalja med z in w.

Lastnosti:

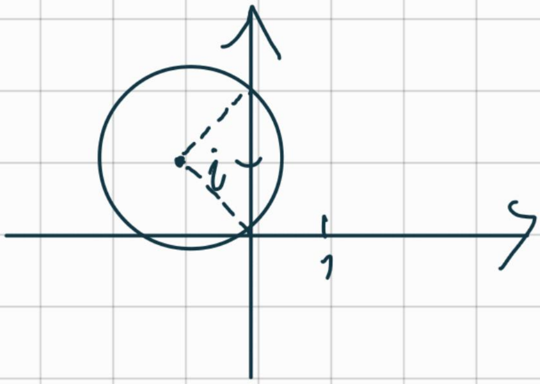
- $|z| \geq 0 \quad \forall z \in \mathbb{C}$
- $|z| = 0 \iff z = 0$
- $|z\cdot w| = |z|\cdot|w| \quad \forall z, w \in \mathbb{C}$
- $|z+w| \leq |z|+|w| \quad \forall z, w \in \mathbb{C}$ (trikotniška neenakost)

Skiciraj v kompleksni ravnini:

- $A = \{ z \in \mathbb{C},\ |z+1-i| = \sqrt{2} \}$

$$|z-(-1+i)| = \sqrt{2}$$

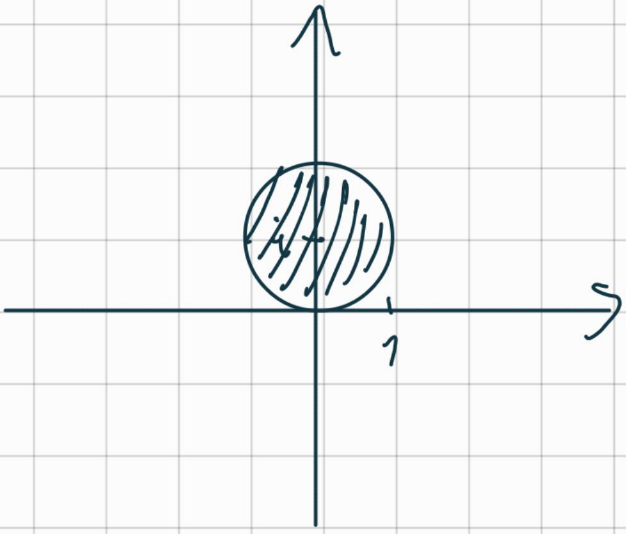
Množica A je krožnica s središčem $-1+i$ in polmerom $\sqrt{2}$.



- $B = \{ z \in \mathbb{C};\ |iz+1| \leq 1 \}$

$$|i(z+\tfrac{1}{i})| \leq 1$$
$$|i(z-i)| \leq 1$$
$$|i| \cdot |z-i| \leq 1$$
$$|z-i| \leq 1$$

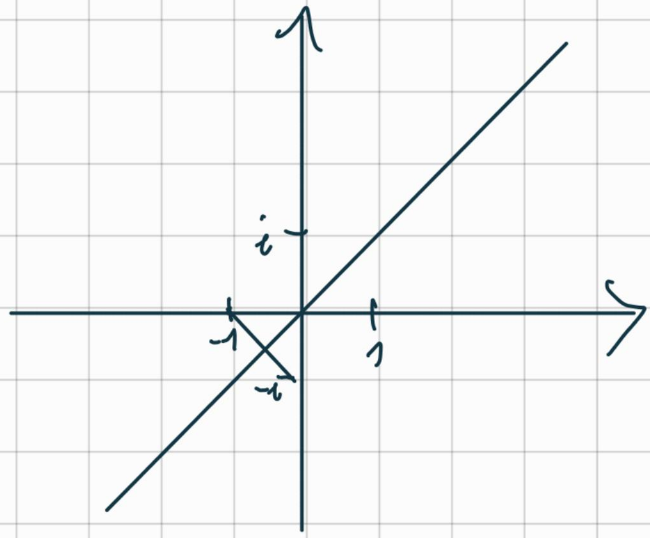
Množica B je zaprt krog s središčem v i in polmerom 1.



- $C = \{ z \in \mathbb{C};\ |z+1| = |z+i| \}$

$$d(z,-1) = d(z,-i)$$

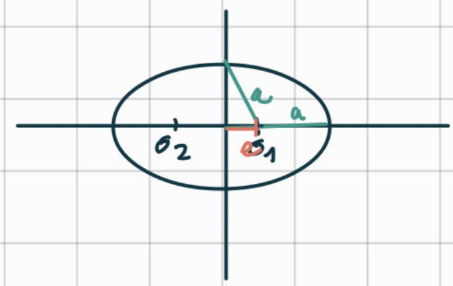
Množica C je simetrala daljice s krajiščema -1 in $-i$.



- $D = \{ z \in \mathbb{C} ;\ |z-4| + |z+3i| = 7 \}$

$$d(z,4) + d(z,-3i) = 7$$

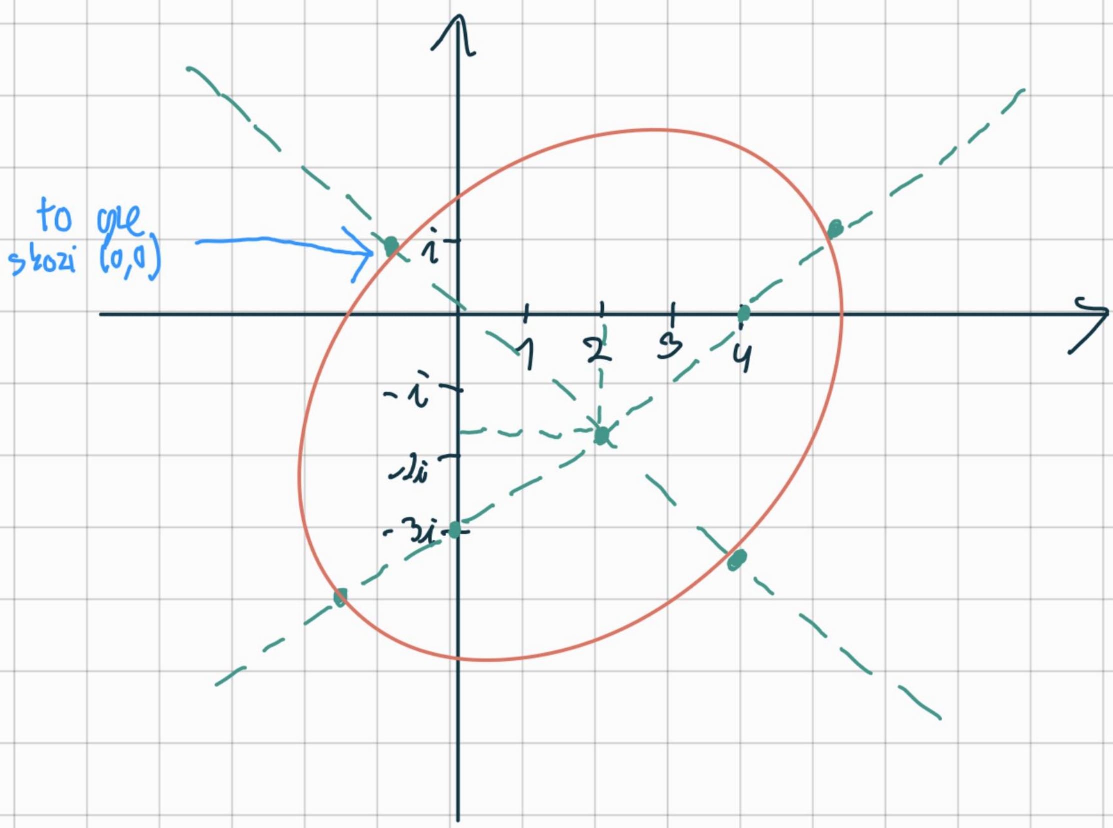
Množica D je elipsa z goriščema 4 in $-3i$.
Središče elipse je $\frac{1}{2}(4-3i)$.
Daljša polos je $a = \frac{1}{2} \cdot 7$.



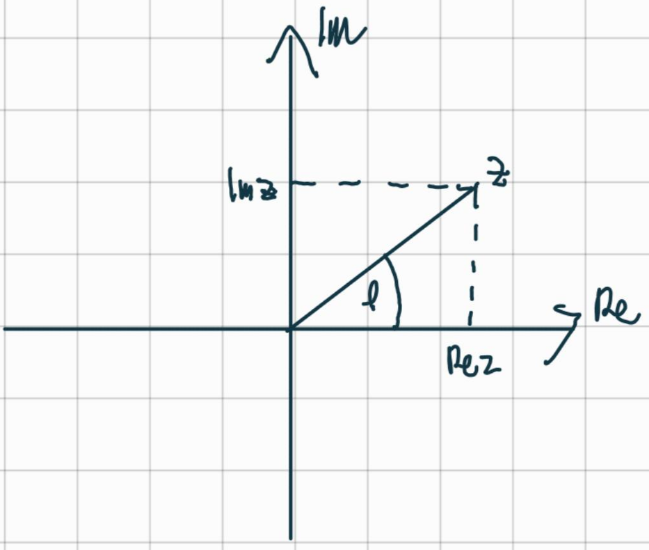
$$e = \frac{1}{2} \cdot d(4, -3i) = \frac{1}{2}\sqrt{16+9} = 5/2$$

$$b = \sqrt{a^2 - e^2} = \sqrt{\frac{49}{4} - \frac{25}{4}} = \sqrt{6}$$

Polosi elipse sta $a = \frac{7}{2}$ in $b = \sqrt{6}$.



POLARNI ZAPIS KOMPLEKSNEGA ŠTEVILA



$x = r\cos\varphi$

$y = r\sin\varphi$

$z = r(\cos\varphi + i\sin\varphi)$

$z = re^{i\varphi}$

$z^n = r^n \cdot (\cos(n\varphi) + i\sin(n\varphi)) \quad \forall n \in \mathbb{N}$

Argument φ ni natančno določen, saj mu lahko prištejemo $2k\pi$, $k \in \mathbb{Z}$ in se z ne spremeni. Včasih se omejimo na $[0, 2\pi)$ ali $(-\pi, \pi]$.

Polarni zapis je uporaben pri množenju in potenciranju (tudi korenjenju) kompleksnih števil.

$z = r(\cos\varphi + i\sin\varphi)$

$w = \rho(\cos\psi + i\sin\psi)$

$z \cdot w = r\rho(\cos\varphi\cos\psi - \sin\varphi\sin\psi + i\cos\varphi\sin\psi + i\sin\varphi\cos\psi)$

$z \cdot w = r\rho(\cos(\varphi+\psi) + i\sin(\varphi+\psi))$

Formule:

$$\boxed{\begin{array}{l} z \cdot w = r\rho(\cos(\varphi+\psi) + i\sin(\varphi+\psi)) \\ z^n = r^n \cdot (\cos(n\varphi) + i\sin(n\varphi)) \quad \forall n \in \mathbb{N} \\ \sqrt[n]{z}_k = \sqrt[n]{r}\left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right), \quad k = 0, 1, \dots, n-1 \end{array}}$$

Primer: Izračunaj $(\sqrt{3} - i)^{12} = \left(2\left(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}\right)\right)^{12}$

$= 2^{12}(\cos 2\pi + i\, 2\pi)$

$= 2^{12}$

Kaj geometrijsko predstavlja preslikava $f: \mathbb{C} \to \mathbb{C}$ s predpisom $f(z) = \alpha \cdot z$, $\alpha \in \mathbb{C}$?

$f(z) = |\alpha| \cdot |z| \cdot (\cos(\varphi + \arg\alpha) + i\sin(\varphi + \arg\alpha))$

f je rotacija za kot $\arg\alpha$ v pozitivno smer in razteg za $|\alpha|$, s središčem v točki 0.

Enačba premice v kompleksnem:

$$\boxed{\alpha z + \overline{\alpha}\,\overline{z} + d = 0} \quad , \quad z \in \mathbb{C}, \quad \alpha \in \mathbb{C}\setminus\{0\}, \quad d \in \mathbb{R}$$

$$ax + by + c = 0$$
$$a \cdot \operatorname{Re} z + b \cdot \operatorname{Im} z + c = 0$$
$$a\,\frac{z+\overline{z}}{2} + b \cdot \frac{\overline{z}-z}{2}\,i + c = 0 \quad / \cdot 2i$$
$$iaz + ia\overline{z} + bz - b\overline{z} + 2ci = 0$$
$$(ia+b)\,z + (ia-b)\,\overline{z} + 2ci = 0 \quad / \cdot i$$
$$\underbrace{(-a+ib)}_{\alpha}\,z + \underbrace{(-a-ib)}_{\overline{\alpha}}\,\overline{z}\ \underbrace{-2c}_{d} = 0$$

$$\alpha z + \overline{\alpha} z + d = 0$$

Enačba krožnice v kompleksnem:

$$|z-\alpha| = r \quad /^2 \qquad z, \alpha \in \mathbb{C}, \quad r \in \mathbb{R}, r > 0$$
$$(z-\alpha)\overline{(z-\alpha)} = r^2$$
$$z\overline{z} - \overline{\alpha} z - \overline{z}\alpha + \alpha\overline{\alpha} = r^2$$
$$z\overline{z} - \overline{\alpha} z - \alpha\overline{z} + |\alpha|^2 - r^2 = 0$$

$$\boxed{z\overline{z} + \beta z + \overline{\beta}\,\overline{z} + d = 0}$$

Opomba: Zgornja enačba nima rešitve za vse d.
Za en d je natanko ena rešitev.

Naloga: Kaj dobimo, če premico zasukamo za kot φ okrog O in jo raztegnemo za faktor $k > 0$?

Vsak z na premici zasukamo za γ in množimo s k.

$$w = z \cdot \underbrace{k \cdot (\cos\gamma + i\sin\gamma)}_{\beta} \Rightarrow z = \frac{w}{\beta}$$

$$\alpha z + \bar{\alpha}\bar{z} + d = 0$$
$$\alpha \frac{w}{\beta} + \bar{\alpha}\left(\overline{\frac{w}{\beta}}\right) + d = 0 \quad / \cdot \beta \cdot \bar{\beta}$$
$$\alpha\bar{\beta} w + \bar{\alpha}\beta\bar{w} + d\beta\bar{\beta} = 0$$

Dobimo premico: $\mu w + \bar{\mu}\bar{w} + e = 0$

Naj bo $n \in \mathbb{N}$. Število w je n-ti koren števila z, če je $w^n = z$. Za vsako neničelno število z obstaja n n-tih korenov.

Naj bo $z = r(\cos\varphi + i\sin\varphi)$, $w = \rho(\cos\psi + i\sin\psi)$. Tedaj je $w^n = z$.

$$\rho^n(\cos(n\psi) + i\sin(n\psi)) = r(\cos\varphi + i\sin\varphi)$$

Dobimo dve enačbi:
$$\rho^n = r \Rightarrow \rho = \sqrt[n]{r}$$
$$n\psi = \varphi + 2k\pi, \quad k \in \mathbb{Z}$$

n-ti koreni števila z so:
$$\boxed{w_k = \sqrt[n]{r}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right), \quad k = 0, 1, \dots, n-1}$$

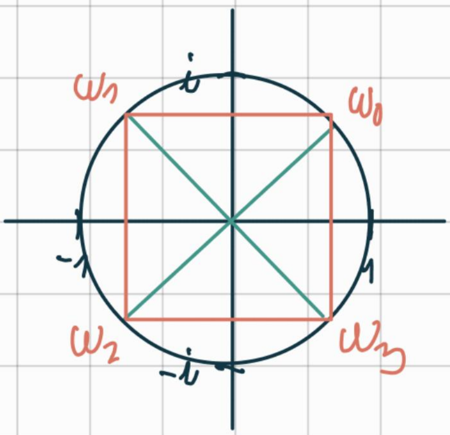
Primer: Izračunaj četrte korene števila -1.

$$z = -1 = \cos\pi + i\sin\pi$$

$$\omega_k = \cos\frac{\pi+2k\pi}{4} + i\sin\frac{\pi+2k\pi}{4}, \quad k=0,1,2,3$$

$$\omega_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$
$$\omega_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$
$$\omega_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$
$$\omega_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

n-ti koreni poljubnega neničelnega števila so oglišča pravilnega n-kotnika s središčem v 0.



Naloga: Reši enačbo $(z-2)^4 + (z+1)^4 = 0$.

$$(z-2)^4 + (z+1)^4 = 0$$
$$(z-2)^4 = -(z+1)^4$$
$$\frac{(z-2)^4}{(z+1)^4} = -1$$
$$\left(\frac{z-2}{z+1}\right)^4 = -1$$
$$\frac{z-2}{z+1} = \omega_k, \quad k=0,1,2,3$$

$$z-2 = z\cdot\omega_k + 1$$
$$z(1-\omega_k) = 2+\omega_k$$
$$z = \frac{2+\omega_k}{1-\omega_k}$$

$$(z-2)^4 + (z+1)^4 = 0$$
$$(z-2)^4 - i(z+1)^4 = 0$$
$$\left((z-2)^2 + i(z+1)^2\right)\left((z-2)^2 - i(z+1)^2\right) = 0$$

1) $(z-2)^2 + i(z+1)^2 = 0$
$$z^2 - 4z + 4 + iz^2 + 2iz + i = 0$$
$$(1+i)z^2 + (-4+2i)z + 4+i = 0$$

$$D = (-4+2i)^2 - 4(1+i)(4+i)$$
$$D = 16 - 16i - 4 - 4(4 + i + 4i - 1)$$
$$D = 12 - 16i - 12 - 20i$$
$$D = -36i$$
$$D = -36\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

$$\sqrt{D} = 6\left(\cos\frac{-\frac{\pi}{2}+2k\pi}{2} + i\sin\frac{-\frac{\pi}{2}+2k\pi}{2}\right),\ k = 0,1$$
$$k=0:\ 6\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 6\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 3\sqrt{2} - i3\sqrt{2}$$
$$k=1:\ -3\sqrt{2} + i3\sqrt{2}$$

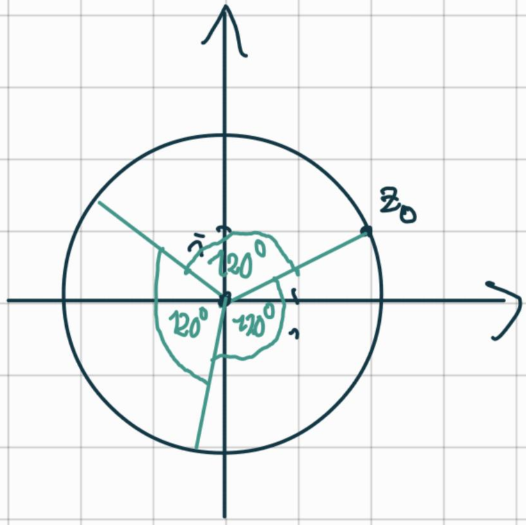
$$\Rightarrow z_{1,2} = \frac{4-2i \pm (3\sqrt{2} - i3\sqrt{2})}{2(1+i)}$$

$$z_1 = \frac{4 + 3\sqrt{2} - 2i - 3\sqrt{2}i}{2(1+i)} \cdot \frac{(1-i)}{(1-i)}$$
$$z_1 = \frac{4 + 3\sqrt{2} - 2i - 3\sqrt{2}i - 4i - 3\sqrt{2} - 2 - 3\sqrt{2}}{4}$$
$$z_1 = \frac{2 - 6i - 6\sqrt{2}i}{4}$$

$$z_2 = \frac{4 - 3\sqrt{2} - 2i + 3\sqrt{2}i}{2(1+i)} \cdot \frac{(1-i)}{(1-i)}$$
$$z_2 = \frac{4 - 3\sqrt{2} - 2i + 3\sqrt{2}i - 4i + 3\sqrt{2}i - 2 + 3\sqrt{2}}{4}$$
$$z_2 = \frac{2 - 6i + 6\sqrt{2}i}{4}$$

2) $(z-2)^2 - i(z+1)^2 = 0$
podobno kot 1)

Naloga: Določi oglišča enakostraničnega trikotnika s središčem v 0, ki ima $2+i$ za oglišče.



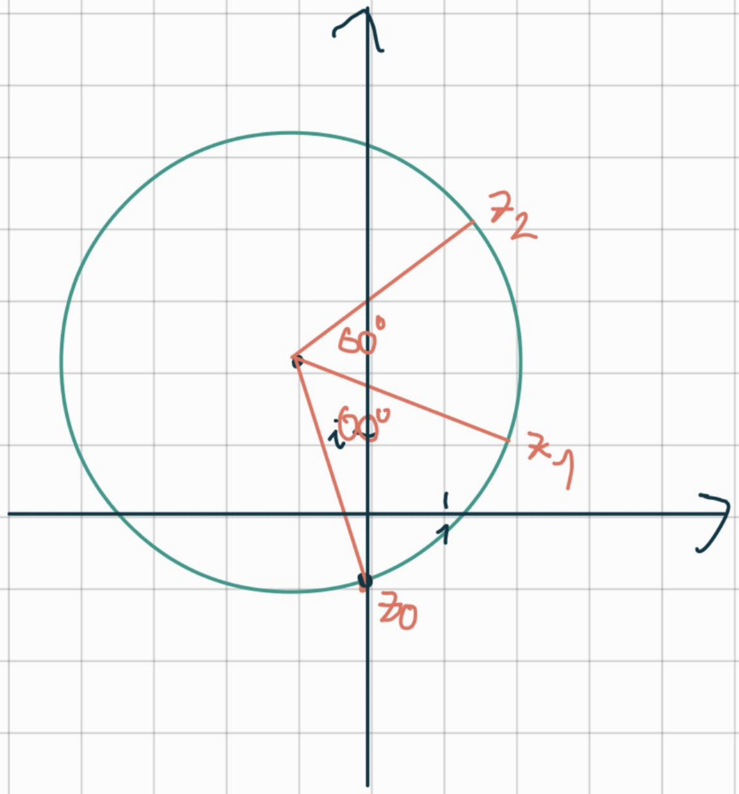
Plan A: Oglišča so tretji koreni števila $(2+i)^3$.

Plan B: Število $2+i$ zavrtimo za kota $\frac{2\pi}{3}$ in $\frac{4\pi}{3}$ $(120°, 240°)$.
To dosežemo tako, da $z_0 = 2+i$ množimo s številoma $\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}$, $k=1,2$.

$$z_1 = z_0\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$
$$z_1 = (2+i)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$z_1 = -1 + i\sqrt{3} - \frac{1}{2}i - \frac{\sqrt{3}}{2}$$
$$z_1 = -1 - \frac{\sqrt{3}}{2} + \left(\sqrt{3} - \frac{1}{2}\right)i$$

$$z_3 = z_0\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$
$$z_3 = (2+i)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$
$$z_3 = -1 - i\sqrt{3} - \frac{1}{2}i + \frac{\sqrt{3}}{2}$$
$$z_3 = -1 + \frac{\sqrt{3}}{2} + \left(-\sqrt{3} - \frac{1}{2}\right)i$$

Naloga: Določi oglišča pravilnega šestkotnika v točki s središčem v $-1+2i$, ki ima $-i$ za oglišče.



Točko z_0 transliramo za $-(-1+2i)$, zarotiramo za $\frac{2k\pi}{6}$, $k=1,2,3,4,5$, nato transliramo za $-1+2i$.

$$z_k = (z_0 - (-1+2i))\left(\cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}\right) + (-1+2i)$$

Naloga: Za vsak $\alpha \in \mathbb{C}\setminus\mathbb{R}$ poišči vse $z \in \mathbb{C}$, da je $|z| = z - \alpha$.

$z = x + iy$
$\alpha = a + ib$

$|z| = z - \alpha$
$\sqrt{x^2+y^2} = x + iy - a - ib$

Re: $\sqrt{x^2+y^2} = x - a \Rightarrow \operatorname{Re} z \geq \operatorname{Re}\alpha$
Im: $0 = y - b \Rightarrow \operatorname{Im} z = \operatorname{Im}\alpha$

$\sqrt{x^2+y^2} = x - a$
$x^2 + y^2 = (x-a)^2$ (na koncu moramo preveriti $x - a \geq 0$)
$x^2 + y^2 = x^2 - 2ax + a^2$
$2ax = a^2 - y^2 = a^2 - b^2$

če $a = 0$, je $b = 0$, zato $\alpha \in \mathbb{R}$, kar ni ok
torej $a \neq 0$

$$x = \frac{a^2 - b^2}{2a}$$

Preverimo pogoj $x - a \geq 0$:

$0 \leq x - a$

$0 \leq \frac{a^2-b^2}{2a} - a$

$0 \leq \frac{a^2-b^2-2a^2}{2a}$

$0 \leq -\frac{a^2+b^2}{2a}$

$\Rightarrow a < 0$

Rešitev za a:

- za $a \geq 0$ ni rešitev
- za $a < 0$ je rešitev $z = \frac{a^2-b^2}{2a} + ib$

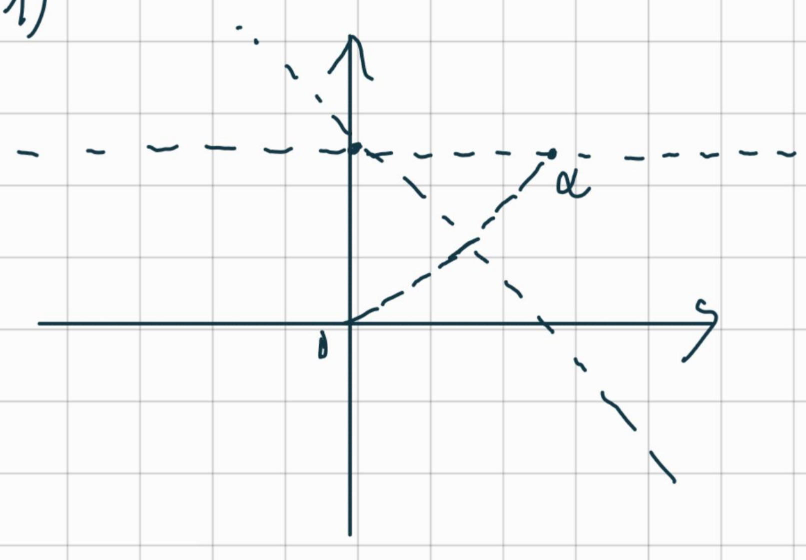
Z geometrijskim premislekom:

$x - a \geq 0$ in $y = b$

$\Rightarrow z - \alpha = |z - \alpha|$

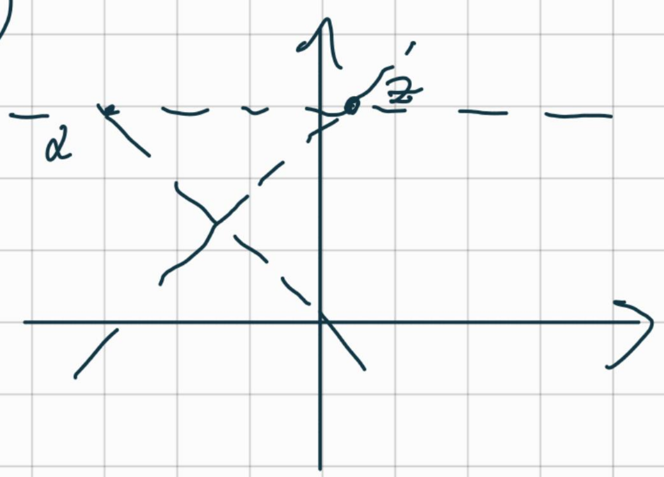
Rešujemo enačbo $|z| = |z - \alpha|$.

1)



ni dobro, ker $\mathrm{Re}\, z < \mathrm{Re}\, \alpha$

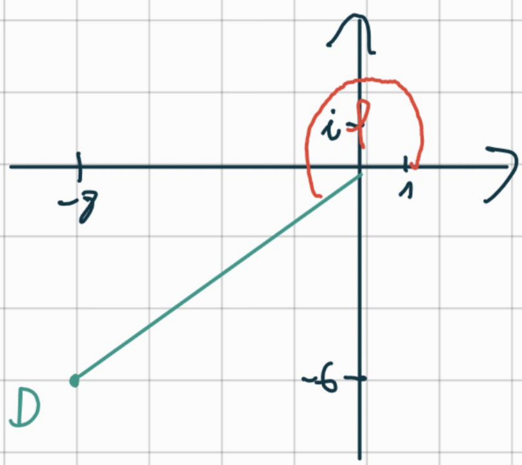
2)



Naloga: Reši enačbo $z^2 - 5(1-i)z + 2 - 11i = 0$.

$$z_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$$ — oba korena števila D

$$D = \left(-5(1-i)\right)^2 - 4(2-11i)$$
$$= 25(1-2i-1) - 8 + 44i$$
$$= -8 - 6i$$



$$D = r(\cos\varphi + i\sin\varphi)$$

$$r = \sqrt{8^2+6^2}$$
$$= 10$$

$$D = 10\Big(\underset{\cos\varphi}{-\tfrac{8}{10}} - \underset{\sin\varphi}{\tfrac{6}{10}}\,i\Big)$$ (tukaj smo samo izpostavili 10)

$$\sqrt{D} = \sqrt{10}\left(\cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}\right)$$

$$\cos^2\varphi + \sin^2\varphi = 1$$
$$\cos^2\varphi - \sin^2\varphi = \cos 2\varphi$$

$$+:\ 2\cos^2\varphi = 1 + \cos 2\varphi$$

$\cos^2\varphi = \frac{1}{2}(1+\cos 2\varphi)$
$\sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi)$

V našem primeru ...

$$\cos\frac{\varphi}{2} = -\sqrt{\frac{1}{2}(1+\cos\varphi)} \quad \left(-, \text{ ker } \frac{\varphi}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\right)$$
$$= -\sqrt{\frac{1}{2}\left(1-\frac{8}{10}\right)}$$
$$= -\frac{1}{\sqrt{10}}$$

$$\sin\frac{\varphi}{2} = +\sqrt{\frac{1}{2}(1-\cos\varphi)}$$
$$= \sqrt{\frac{1}{2}\left(1-\frac{8}{10}\right)}$$
$$= \frac{3}{\sqrt{10}}$$

$$\sqrt{D} = \pm\sqrt{10}\left(-\frac{1}{\sqrt{10}} + \frac{3}{\sqrt{10}}i\right)$$
$$= \pm(-1+3i)$$

$$z_{1,2} = \frac{5-5i \pm (-1+3i)}{2}$$

$$z_1 = \frac{5-5i-1+3i}{2} = \frac{4-2i}{2} = 2-i$$
$$z_2 = \frac{5-5i+1-3i}{2} = \frac{6-8i}{2} = 3-4i$$

Plan B:

$$\sqrt{D} = x+yi \quad /^2$$
$$D = x^2+2xyi-y^2$$

$$-8-6i = x^2+2xyi-y^2$$

Re: $-8 = x^2-y^2$
Im: $-6 = 2xy \Rightarrow y = -\frac{3}{x}$

$$-8 = x^2 - \frac{9}{x^2}$$

$$x^4 + 8x^2 - 9 = 0$$

$$(x^2+9)(x^2-1) = 0$$

$$x_{1,2} = \pm 1$$

$$y_{1,2} = \mp 3$$

$z = r(\cos\varphi + i\sin\varphi)$, iščemo $\sqrt[n]{z}$ ...

$$w_k = \sqrt[n]{r}\left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right)$$

$k = 0, 1, \dots, n-1$