V vektorski prostor s skalarnim produktom
$\dim V < \infty$

$\mathcal{A}: V \to V$ linearna preslikava

Definicija: $\mathcal{A}$ je normalna $\iff \mathcal{A}\mathcal{A}^* = \mathcal{A}^*\mathcal{A}$

Izrek: V ima ONB iz lastnih vektorjev $\mathcal{A}$
($\mathcal{A}$ se diagonalizira glede na neko ONB prostora V)
$\iff \mathcal{A}$ je normalna in vse lastne vrednosti so v $\mathbb{F}$

Lastnosti:

- Lastni vektorji za različne lastne vrednosti so pravokotni.
- $\|\mathcal{A}x\| = \|\mathcal{A}^*x\|$ za vsak $x \in V$
- $\ker \mathcal{A}^* = \ker \mathcal{A}$

$A \in \mathbb{F}^{n \times n}$

Definicija: A je normalna $\iff \underset{(\mathbb{F}=\mathbb{C})}{A^H A = AA^H}$ oz. $\underset{(\mathbb{F}=\mathbb{R})}{A^T A = AA^T}$

Posebne vrste normalnih preslikav:

- sebi adjungirane: $\mathcal{A}^* = \mathcal{A}$

$\mathcal{A}^* = \mathcal{A} \iff \mathcal{A}$ je normalna in ima realne last. vred.

$A^H = A \Leftrightarrow A$ je unitarno podobna realni diagonalni matriki (za $A \in \mathbb{C}^{n \times n}$)

$A^T = A \Leftrightarrow A$ je ortogonalno podobna realni diagonalni matriki (za $A \in \mathbb{R}^{n \times n}$)

- unitarne: $\mathcal{A}^* \mathcal{A} = \mathcal{A}\mathcal{A}^* = id$

$\mathcal{A}$ je unitarna $\Leftrightarrow$ $\mathcal{A}$ je normalna in ima vse lastne vrednosti na enotski krožnici $(|\lambda| = 1)$

$\mathcal{A}$ je unitarna $\Leftrightarrow \|\mathcal{A}x\| = \|x\| \quad \forall x \in V$

$\mathcal{A}$ je unitarna $\Leftrightarrow \langle \mathcal{A}x, \mathcal{A}y \rangle = \langle x, y \rangle \quad \forall x, y \in V$

$A \in \mathbb{C}^{n \times n}$ je unitarna, če je $A^H A = A A^H = I$.
$A \in \mathbb{R}^{n \times n}$ je ortogonalna, če je $A^T A = A A^T = I$.

- pozitivno (semi) definitne:

$\mathbb{F} = \mathbb{C}$: $\mathcal{A}$ je PSD, če $\langle \mathcal{A}x, x \rangle \geq 0 \quad \forall x \in V$.
$\mathbb{F} = \mathbb{R}$: $\mathcal{A}$ je PSD, če –||– in $\mathcal{A}^* = \mathcal{A}$.

$\mathcal{A}$ je PD, če je PSD in $\langle \mathcal{A}x, x \rangle > 0 \quad \forall x \in V$.

$\mathcal{A}$ je PD $\Leftrightarrow$ $\mathcal{A}$ je PSD in obrnljiva

$\mathcal{A}$ je PSD $\Leftrightarrow$ $\mathcal{A}$ je normalna in ima vse lastne vrednosti nenegativne

$\mathcal{A}$ je PD $\Leftrightarrow$ $\mathcal{A}$ je normalna in ima vse lastne vrednosti pozitivne

$A \in \mathbb{C}^{n\times n}$ je PSD $\Leftrightarrow$ $A = A^H$ in vsi glavni minorji nenegativni

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1) Dokaži, da predpis:

$$[(x,y,z),(u,v,w)] = 2xu - yu - xv + 2yv - zv - yw + zw$$

predstavlja skalarni produkt na $\mathbb{R}^3$. Naj bo:

$$A = \begin{bmatrix} 0 & 2 & -2 \\ 0 & 1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$$

matrika endomorfizma $\mathcal{A}$ na $\mathbb{R}^3$ glede na stand. bazo.
Ali je $\mathcal{A}$ normalen glede na $[\,;\,]$?

Linearnost v prvem faktorju:

DN

Simetričnost:

DN

Pozitivna definitnost:

$$[(x,y,z),(x,y,z)] = 2x^2 - yx - xy + 2y^2 - zy - yz + z^2 =$$
$$= 2x^2 - 2xy + 2y^2 - 2yz + z^2 =$$
$$= (x-y)^2 + x^2 + (y-z)^2 \geq 0 \quad \checkmark$$

$$[(x,y,z),(x,y,z)] = 0 \Leftrightarrow x = y = z = 0 \quad \checkmark$$

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 2 & -2 \\ 0 & 1-\lambda & 0 \\ -1 & 2 & -1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & -2 \\ -1 & -1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\big((-\lambda)(-1-\lambda)-2\big)$$

$\lambda_1 = \lambda_2 = 1$
$\lambda_3 = -2$

$\lambda = 1$:

$$\begin{bmatrix} -1 & 2 & -2 \\ 0 & 0 & 0 \\ -1 & 2 & -2 \end{bmatrix}$$

$v_1 = (2, 0, -1)$
$v_2 = (0, 1, 1)$

$\lambda = -2$:

$$\begin{bmatrix} 2 & 2 & -2 \\ 0 & 3 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

$v_3 = (1, 0, 1)$

Vse lastne vrednosti so realne.

$[v_1, v_3] = \dots$
$[v_2, v_3] = \dots = -2 \neq 0$

Lastna vektorja za različni lastni vrednosti nista pravokotna, torej A ni normalen.

2) V vek. prostor s skal. produktom, $\dim V < \infty$
$\mathcal{A}, \mathcal{B}: V \to V$ normalna endomorfizma
$\mathcal{A}\mathcal{B} = 0$

Dokaži: $\mathcal{B}\mathcal{A} = 0$

$$\mathcal{A}\mathcal{B} = 0 \Rightarrow \operatorname{im}\mathcal{B} \subseteq \ker\mathcal{A} \quad /^{\perp}$$
$$(\operatorname{im}\mathcal{B})^{\perp} \supseteq (\ker\mathcal{A})^{\perp}$$
$$\ker\mathcal{B} = \ker\mathcal{B}^* \supseteq \operatorname{im}\mathcal{A}^*$$

$\mathcal{B}$ normalen

$$\ker\mathcal{A} = \ker\mathcal{A}^*$$
$$(\ker\mathcal{A})^{\perp} = (\ker\mathcal{A}^*)^{\perp} = \operatorname{im}\mathcal{A}^{**} = \operatorname{im}\mathcal{A}$$

$$\Rightarrow \ker\mathcal{B} \supseteq \operatorname{im}\mathcal{A}$$
$$\Rightarrow \mathcal{B}\mathcal{A} = 0$$

Primer ne normalnih matrik, da $AB = 0$ in $BA \neq 0$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

ni normalna normalna

$$AB = 0$$
$$BA \neq 0$$

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3) Naj bosta $B, C \in \mathbb{R}^{n \times n}$. Dokaži, da je matrika

$$D = \begin{bmatrix} B & -C \\ C & B \end{bmatrix}$$

normalna natanko tedaj, ko je $B + iC$ normalna.

$$DD^T = \begin{bmatrix} B & -C \\ C & B \end{bmatrix}\begin{bmatrix} B^T & C^T \\ -C^T & B^T \end{bmatrix} = \begin{bmatrix} BB^T + CC^T & BC^T - CB^T \\ CB^T - BC^T & CC^T + BB^T \end{bmatrix}$$

$$D^TD = \begin{bmatrix} B^T & C^T \\ -C^T & B^T \end{bmatrix}\begin{bmatrix} B & -C \\ C & B \end{bmatrix} = \begin{bmatrix} B^TB + C^TC & -B^TC + C^TB \\ -C^TB + B^TC & C^TC + B^TB \end{bmatrix}$$

D je normalna $\iff BB^T + CC^T = B^TB + C^TC$
in $BC^T - CB^T = C^TB - B^TC$

$$(B+iC)\cdot(B+iC)^H = (B+iC)(B^T - iC^T) =$$
$$= BB^T - iBC^T + iCB^T + CC^T$$

$$(B+iC)^H \cdot (B+iC) = (B^T - iC)(B+iC) =$$
$$= B^TB + iB^TC - iC^TB + C^TC$$

$(B+iC)$ je normalna $\iff BB^T + CC^T = B^TB + C^TC$
in $-BC^T + CB^T = -C^TB + B^TC$

Pogoja sta ekvivalentna.

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4) Naj bo $a \in \mathbb{C}^n$, $\|a\| = 1$. Za katere $\alpha \in \mathbb{C}$ je matrika $A = I - \alpha a a^H$:

a) unitarna

$$A^H = (I - \alpha a a^H)^H = I - \overline{\alpha} a^{H^H} a^H = I - \overline{\alpha} a a^H$$

$$a^H a = [\overline{a}_1 \cdots \overline{a}_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = [a_1\overline{a}_1 + \ldots + a_n\overline{a}_n] = 1$$

$x^Hy = \langle x, x \rangle$, za standardni $\langle \cdot, \cdot \rangle$

$$AA^H = (I - \alpha aa^H)(I - \bar{\alpha} aa^H) =$$
$$= I^2 - \alpha aa^H - \bar{\alpha} aa^H + \alpha\bar{\alpha} aa^H aa^H =$$
$$= I - aa^H(\alpha + \bar{\alpha}) + |\alpha|^2 aa^H =$$
$$= \underset{\neq 0}{I - aa^H} (\alpha + \bar{\alpha} - |\alpha|^2) = I$$

$\Leftrightarrow \alpha + \bar{\alpha} - |\alpha|^2 = 0$, $\alpha = x + yi$

$x + yi + x - yi + x^2 + y^2 = 0$

$x^2 - 2x_i y + y^2 = 0$

$(x-1)^2 + y^2 = 1$

$|\alpha - 1| = 1$

krožnica s središčem v 1 in polmerom 1

$xy^T = 0 \Leftrightarrow x = 0$ ali $y = 0$

b) hermitska

$$A^H = (I - \alpha aa^H)^H = I - \bar{\alpha} a^{H^H} a^H = I - \bar{\alpha} aa^H$$

A hermitska $\Leftrightarrow \alpha \in \mathbb{R}$

c) hermitska in unitarna

$\alpha = 0$ ali $\alpha = 2$

$\downarrow$ I $\quad$ $\downarrow$ zrcaljenje

2. način:

$$Ax = x - \alpha \overbrace{\langle x, a \rangle}^{a^H x} a$$

i) $x \perp a \Rightarrow Ax = x \Rightarrow \lambda = 1$

$n-1$ lin. neod. last. vek.

ii) $Aa = (1-\alpha) a \Rightarrow \lambda = 1 - \alpha$

1 lin. neod. last. vek.

Preverimo: $AA^H = A^H A$

A je unitarna $\Leftrightarrow |1-\alpha| = 1$
A je hermitska $\Leftrightarrow 1-\alpha \in \mathbb{R} \Leftrightarrow \alpha \in \mathbb{R}$

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5) $A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{bmatrix}$

Poišči P, da je $P^T A P$ diagonalna.

P je ortogonalna matrika.

Lastne vrednosti:
$\lambda_{1,2} = 3$
$\lambda_3 = 6$

$\lambda = 6$:

$$A - 6I = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$v_1' = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\qquad v_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

$\lambda = 3$:

$v_2' = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\qquad v_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$$v_3' = v_1 \times v_2 = \begin{bmatrix} \frac{2}{\sqrt{76}} \\ \frac{1}{\sqrt{76}} \\ \frac{-1}{\sqrt{76}} \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & \frac{2}{\sqrt{76}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{76}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{76}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

7a) Ali obstaja ortogonalna $P \in \mathbb{R}^{n \times n}$, da je $P^T = -P$?

$n = 1$:

$$P^T = P = -P$$
$$P = 0$$

To ni ortogonalna 1×1 matrika.

$n = 2$:

$$P = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

Če je $a \in \{-1, 1\}$, je to ortogonalna matrika.

$n = 2k$:

$$P = \begin{array}{cc} \begin{array}{cc} \quad k \quad & \quad k \quad \end{array} & \\ \left[\begin{array}{c|c} & \begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix} \\ \hline \begin{matrix} & & -1 \\ & \iddots & \\ -1 & & \end{matrix} & \end{array}\right] & \begin{array}{c} k \\ \\ k \end{array} \end{array}$$

Za $n = 2k$ obstaja.

$n = 2k+1$:

Če je P ortogonalna, ima lastne vrednosti na enotski krožnici.

Naj bo λ lastna vrednost in $x \in \mathbb{C}^n$ njen enotski lastni vektor: $Px = \lambda x$.

$$\langle Px, x \rangle = \langle x, P^T x \rangle = \langle x, -Px \rangle$$
$$\langle \lambda x, x \rangle = \langle x, -\lambda x \rangle$$
$$\lambda = -\overline{\lambda}$$

$\Rightarrow \lambda = i \cdot y, \ y \in \mathbb{R}$
$\Rightarrow y = \pm i$ (da je $|\lambda| = 1$)

Ker je P realna matrika, ima njen karakteristični polinom realne koeficiente.

Ker je lihe stopnje, ima vsaj eno realno ničlo.
$\times$ (ker so vse ničle $\lambda = \pm i$)

7b) Ali obstaja unitarna $U \in \mathbb{C}^{n \times n}$, da je $U^H = -U$?

$n = 1$:

$z \in \mathbb{C}$
$\overline{z} = -z$
$|z| = 1$

$\Rightarrow z = \pm i$

$n \geq 1$:

$$U = \begin{bmatrix} \pm i & & \\ & \ddots & \\ & & \pm i \end{bmatrix}$$

Obstaja za vsak n.

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8) V unitarni prostor, $W \leq V$ podprostor.
$\mathcal{U}: V \to V$ unitarni endomorfizem.
Dokaži: $\mathcal{U}(W)^\perp = \mathcal{U}(W^\perp)$

$\langle \mathcal{U}x, \mathcal{U}y \rangle = \langle x, \mathcal{U}^*\mathcal{U}y \rangle = \langle x, y \rangle$
($\mathcal{U}$ ohranja skalarni produkt)

$\mathcal{U}(W)^\perp \subseteq \mathcal{U}(W^\perp)$

$x \in \mathcal{U}(W)^\perp$
$x = \mathcal{U}y$ (ker je $\mathcal{U}$ bijekcija)

$y \in W^\perp \Leftrightarrow \forall w \in W.\ \langle y, w \rangle = 0$

$$\langle y, w \rangle = \langle \mathcal{U}y, \mathcal{U}w \rangle = \langle \underset{\mathcal{U}(W)^\perp}{\underset{\Uparrow}{x}}, \underset{\mathcal{U}(W)}{\underset{\Uparrow}{\mathcal{U}w}} \rangle = 0$$

$y \in W^\perp$

$\mathcal{U}(W^\perp) \subseteq \mathcal{U}(W)^\perp$

$x \in \mathcal{U}(W^\perp)$
$x = \mathcal{U}y,\ y \in W^\perp$

$x \in \mathcal{U}(W)^\perp \Leftrightarrow \forall z \in \mathcal{U}(W).\ \langle x, z \rangle = 0$

$z = \mathcal{U}w,\ w \in W$

$\langle x, z \rangle = \langle \mathcal{U}y, \mathcal{U}w \rangle = \langle y, w \rangle = 0$

$z \in \mathcal{U}(W)^\perp$

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9) Naj bo $A \in \mathbb{C}^{n \times n}$. Dokaži, da je A normalna natanko tedaj, ko je $AA^H - A^HA$ pozitivno semidefinitna.

B je PSD $\Leftrightarrow B = B^H$ in $\forall x \in \mathbb{C}^n: \langle Bx, x \rangle \geq 0$

($\Rightarrow$) A je normalna $\Rightarrow AA^H = A^HA \Rightarrow$
$\Rightarrow AA^H - A^HA = 0 \geq 0 \Rightarrow AA^H - A^HA$ je PSD

($\Leftarrow$) $\underbrace{AA^H - A^HA}_{= B}$ je PSD

$(AA^H - A^HA)^H = A^{HH}A^H - A^HA^{HH} = AA^H - A^HA$
$\Rightarrow$ je sebi adjungirana

$x \in \mathbb{C}^n$

$\langle Bx, x \rangle \geq 0$

$B = 0$

$B = B^H \Rightarrow B$ je normalen $\Rightarrow B$ se diagonalizira v ONB $\Rightarrow \lambda_1, \dots, \lambda_n$ imajo pripadajoče lastne vektorje $x_1, \dots, x_n$

$$\langle Bx_k, x_k \rangle = \langle \lambda_k x_k, x_k \rangle = \lambda_k \overbrace{\langle x_k, x_k \rangle}^{=1} \geq 0$$
$$\Rightarrow \lambda_k \geq 0$$

$$\operatorname{tr} B = \sum_{i=1}^{n} \lambda_i = \operatorname{tr} AA^H - \operatorname{tr} A^H A =$$
$$= \operatorname{tr} AA^H - \operatorname{tr} AA^H = 0$$

$$\operatorname{tr} XY = \operatorname{tr} YX$$

$$\lambda_1 + \dots + \lambda_n = 0$$
$$\lambda_k \geq 0 \quad \forall k$$

$$\Rightarrow \lambda_k = 0$$
$$\Rightarrow B = 0$$

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10) Naj bo $(V, \langle \cdot, \cdot \rangle)$ vektorski prostor.

a) Naj bo $\mathcal{A}: V \to V$ PD. Dokaži, da je $[v, w] = \langle v, \mathcal{A} w \rangle$ skalarni produkt na V.

$[x,x] \geq 0$

$$[x,x] = \langle x, \mathcal{A}x \rangle \overset{\mathcal{A}^* = \mathcal{A}}{=} \langle \mathcal{A}x, x \rangle \overset{x \neq 0}{>} 0 \quad ✓$$

$$[0,0] = \langle 0, \mathcal{A}0 \rangle = 0 \quad ✓$$

$$[x,x] = 0 \Leftrightarrow x = 0 \quad ✓$$

$[x,y] = \overline{[y,x]}$

$$\overline{[y,x]} = \overline{\langle y, \mathcal{A}x \rangle} = \langle \mathcal{A}x, y \rangle = \langle x, \mathcal{A}y \rangle = [x,y]$$

$[\alpha x + \beta y, z] = \alpha [x,z] + \beta [y,z]$

$[\alpha x+\beta y, z] = \langle \alpha x+\beta y, \mathcal{A}z\rangle = \langle \alpha x, \mathcal{A}z\rangle + \langle \beta y, \mathcal{A}z\rangle =$
$= \alpha[x,z] + \beta[y,z]$

b) Dokaži, da za vsak skalarni produkt $[\cdot,\cdot]$ obstaja natanko en PD $\mathcal{A}: V \to V$, da je $[v,w] = \langle v, \mathcal{A}w\rangle$.

$[x,y] = f(x)$ za fiksen y
f je linearen funkcional

natančno določen

To lahko predstavimo kot $f(x) = \langle x, y'\rangle$ za primeren y'.

$[x,y] = \langle x, y'\rangle \overset{\sim}{=} \langle x, \mathcal{A}y\rangle$

$\mathcal{A}y = y'$

$\mathcal{A}$ je linearna

$[x, \alpha y+\beta z] = \underline{\langle x, \mathcal{A}(\alpha y+\beta z)\rangle}$

$[x, \alpha y+\beta z] = [x, \alpha y] + [x, \beta z] = \bar{\alpha}[x,y] + \bar{\beta}[x,z] =$
$= \bar{\alpha}\langle x, \mathcal{A}y\rangle + \bar{\beta}\langle x, \beta z\rangle = \langle x, \alpha\mathcal{A}y\rangle + \langle x, \beta\mathcal{A}z\rangle =$
$= \underline{\langle x, \alpha\mathcal{A}y + \beta\mathcal{A}z\rangle}$

$\Rightarrow \mathcal{A}$ je linearna

$\mathcal{A}$ je PD (glede na $\langle\cdot,\cdot\rangle$)

$\mathcal{A} = \mathcal{A}^*$

$\langle x, \mathcal{A}y\rangle = [x,y] = \overline{[y,x]} = \overline{\langle y, \mathcal{A}x\rangle} = \langle \mathcal{A}y, x\rangle$

$\forall x \neq 0.\ \langle \mathcal{A}x, x\rangle > 0$

$$\langle Ax, x\rangle \overset{A=A^*}{=} \langle x, Ax\rangle = [x,x] \overset{x\neq 0}{>} 0$$

c) S pomočjo (a) se prepričaj, da je
$[(x,y,z),(u,v,w)] = 2xu - yu - xv + 2yv - zv - yw + zw$
skalarni produkt.

Naj bo $\langle \cdot,\cdot \rangle$ standardni skalarni produkt na $\mathbb{R}^3$.
Po točki (b) iščemo A, da bo $[\vec{x},\vec{y}] = \langle \vec{x}, A\vec{y}\rangle$.

$$A\vec{y} = \vec{z}$$

$$\vec{x} = (x,y,z)$$
$$\vec{y} = (u,v,w)$$
$$\vec{z} = (z_1, z_2, z_3)$$

$$[\vec{x},\vec{y}] = \langle \vec{x},\vec{z}\rangle = xz_1 + yz_2 + zz_3 =$$

$$= x(2u-v) + y(2v-u-w) + z(w-v)$$

$$A\vec{y} = \begin{bmatrix} \\ A \\ \\ \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\Rightarrow A = \begin{array}{cc} \begin{array}{ccc} u & v & w \end{array} & \\ \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \end{array}$$

$A > 0 \iff$ glavni minorji so > 0

- $|2| > 0$ ✓

- $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$ ✓

- $\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix} = 1 > 0$ ✓

$\Rightarrow A$ je PD

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2) $A \in \mathbb{C}^{n \times n}$ je PD. Dokaži, da obstaja $\sqrt{A} > 0$.

A ima pozitivne lastne vrednosti in A se diagonalizira v ONB

$$A = U \cdot \overset{\overset{D}{\shortparallel}}{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}} \cdot U^{-1} = UDU^H$$

U unitarna

$$\sqrt{A} = U \cdot \sqrt{D} \cdot U^H$$

$$\sqrt{D} = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix} > 0, \text{ ker } \lambda_i > 0$$

$$(U\sqrt{D}U^H)^2 = (U\sqrt{D}\overbrace{U^H)(U}^{I}\sqrt{D}U^H) = U\overbrace{\sqrt{D}\sqrt{D}}^{D}U^H =$$
$$= UDU^H = A$$

B je PSD $\Leftrightarrow$ se diagonalizira v neki ONB in so lastne vrednosti ≥ 0

U je unitarna $\Leftrightarrow$ stolpci/vrstice so ONB

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11) $A \in \mathbb{C}^{n\times n}$ je PD in $T \in \mathbb{C}^{n\times n}$, da $A^{-1}T^H A$ komutira s T. Dokaži, da se A diagonalizira.

$$(A^{-1}T^H A)T = T(A^{-1}T^H A)$$

$B = \sqrt{A}$ iz prejšnje naloge

$$B\,/\quad B^{-2}T^H B^2 T = T B^{-2} T^H B^2 \quad /\,B^{-1}$$
$$B^{-1}T^H B^2 T B^{-1} = B T B^{-2} T^H B$$
$$\underline{B^{-1}T^H B}\,\underline{B T B^{-1}} = \underline{B T B^{-1}}\,\underline{B^{-1} T^H B}$$

$$= (B^{-1}T^H B)^H = B^H T (B^{-1})^H = B T B^{-1} = \quad = S$$

$B^H = B$
$(X^{-1})^H = (X^H)^{-1}$

$$S^H S = S S^H$$

$\Rightarrow$ S je normalen
$\Rightarrow$ S se diagonalizira

$S = B T B^{-1}$
$\Rightarrow$ $S \sim T$

$\Rightarrow$ T se diagonalizira

Drug način:

$[x,y] = \langle x, \mathcal{A} y\rangle$ je skal. prod, ko je $\mathcal{A}$ PD

T^H pripada $\mathcal{T}^*$ v stand. skal. produktu.

Kaj pripada $\mathcal{T}^*$ v $[\cdot,\cdot]$?

$[Tx, y] = \langle Tx, Ay \rangle = \langle x, \underline{T^H A} y \rangle =$ vrinemo $A^{-1}A$

$= \langle x, A \cdot A^{-1} T^H A y \rangle = [x, A^{-1} T^H A y]$

$\Rightarrow T^{[H]} = A^{-1} T^H A$

Vemo: $(A^{-1} T^H A) T \overset{(*)}{=} T (A^{-1} T^H A)$

$T \cdot T^{[H]} \overset{(*)}{=} T^{[H]} \cdot T$

$\Rightarrow T$ je normalna v skalarnem produktu $[\cdot ; \cdot]$

$\Rightarrow T$ se diagonalizira

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12) Naj bo $A \in \mathbb{C}^{n \times n}$.

a) Dokaži: A je normalna $\Leftrightarrow$ $A^H = p(A)$ za nek polinom p

($\Rightarrow$) A je normalna: $A^H A = A A^H$

$\Rightarrow A = U \cdot D \cdot U^H$
za unitarno U in diagonalno D

$A^H = (U D U^H)^H = U D^H U^H$

$p(A) = p(U D U^H) = U \cdot p(D) \cdot U^H$

Želimo si: $U D^H U^H = U p(D) U^H$
Torej: $D^H = p(D)$

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad D^H = \begin{bmatrix} \overline{\lambda_1} & & \\ & \ddots & \\ & & \overline{\lambda_n} \end{bmatrix}$$

$$p(D) = \begin{bmatrix} p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_n) \end{bmatrix}$$

Iščemo p, da bo $p(\lambda_k) = \overline{\lambda_k} \quad \forall k$

$\mu_1, \dots, \mu_r \in \mathbb{C}$ različne točke
$\nu_1, \dots, \nu_r \in \mathbb{C}$ poljubna števila

Vemo: $\exists p \in \mathbb{C}_r[\mu] : \; p(\mu_i) = \nu_i$

Vzamemo različne lastne vrednosti λ_i.
Definiramo $p(\lambda_i) = \overline{\lambda_i}$.
To določa naš polinom.

$(\Leftarrow)$ $A^H = p(A)$ za nek polinom p

$$A^H A = p(A) \cdot A \underset{\substack{\text{matrika komutira} \\ \text{s funkcijo v matriki}}}{=} A \cdot p(A) = A A^H$$

b) Naj bo A normalna in $B \in \mathbb{C}^{n \times n}$, ki komutira z A. Dokaži, da B komutira z A^H.

$\underline{BA^H = A^H B}$

$$A^HA = AA^H \overset{(a)}{\Longrightarrow} A^H = p(A)$$

$$BA^H \overset{(a)}{=} B\cdot p(A) = B\cdot \sum_{j=0}^{k} a_j A^j \overset{BA=AB}{=} \left(\sum_{j=0}^{k} a_j A^j\right) B =$$

B dela hop hop hop

$$= p(A)\cdot B \overset{(a)}{=} A^H B \quad ✓$$

c) Primer nenormalne matrike, kjer to ne velja.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ni normalna, ker se ne diagonalizira

Iščemo B, da $AB = BA$, ampak $A^HB \neq BA^H$.

Vzamemo $B = A$.

$AB = BA$, ker $B = A$.

Ampak $A^HB \neq BA^H$, ker $A^HA \neq AA^H$, ker A ni normalna.

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13) Naj bo $H = [h_{ij}]_{i,j=1}^{n}$ hermitska matrika in λ njena največja lastna vrednost. Dokaži, da velja $\max\limits_{i=1,\dots,n} h_{ii} \leq \lambda$.

Ker je H hermitska, so njene lastne vrednosti in diagonalni elementi realni.

He_i ... i-ti stolpec

$\Rightarrow \langle He_i, e_i \rangle = h_{ii}$

$H = H^H$

$\Rightarrow \exists\, U$ unitarna, D diagonalna: $H = UDU^H$

$$h_{ii} = \langle He_i, e_i\rangle = \langle UDU^He_i, e_i\rangle = \langle D\underline{U^He_i}, \underline{U^He_i}\rangle$$

$y_i := U^He_i$

U unitarna

$\Rightarrow U^H$ unitarna

$\Rightarrow \|y_i\| = \|e_i\| = 1$

$$h_{ii} = \langle D y_i, y_i \rangle = \left\langle \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} y_{i1} \\ \vdots \\ y_{in} \end{bmatrix}, \begin{bmatrix} y_{i1} \\ \vdots \\ y_{in} \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \lambda_1 y_{i1} \\ \vdots \\ \lambda_n y_{in} \end{bmatrix}, \begin{bmatrix} y_{i1} \\ \vdots \\ y_{in} \end{bmatrix} \right\rangle =$$

$$= \sum_{j=1}^{n} \lambda_j y_{ij} \overline{y_{ij}} = \sum_{j=1}^{n} \lambda_j \overset{0\,\leq}{|y_{ij}|^2} \leq \sum_{j=1}^{n} \lambda |y_{ij}|^2 = \lambda \|y_i\|^2 = \lambda$$