Naj bo $A \in \mathbb{C}^{n \times n}$ matrika.

Naj bo f funkcija, ki je $(n-1)$-krat odvedljiva v vsaki lastni vrednosti matrike A.

$$A = P \cdot J(A) \cdot P^{-1}$$

$$f(A) = P \cdot f(J(A)) \cdot P^{-1}$$

$$J(A) = \begin{bmatrix} \ddots & & \\ & \Lambda & \\ & & \ddots \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$$

$$f(J(A)) = \begin{bmatrix} \ddots & & \\ & f(\Lambda) & \\ & & \ddots \end{bmatrix}$$

$$f(\Lambda) = \begin{bmatrix} f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2!} & \cdots & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ & \ddots & & & \vdots \\ & & \ddots & & \frac{f''(\lambda)}{2!} \\ & & & \ddots & f'(\lambda) \\ & & & & f(\lambda) \end{bmatrix}$$

---

1) Izračunaj e^A za matriko A.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\Delta_A(\lambda) = (\lambda-1)^2(\lambda-2) \qquad (DN)$$

$$J(A) = \left[\begin{array}{cc|c} 1 & 1 & \\ & 1 & \\ \hline & & 2 \end{array}\right]$$

$\lambda = 1:$

$$B = A - I = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \end{array}$$

$\operatorname{rang} B = 2$
$\Rightarrow d_1 = 1$

Število kletk je število lastnih vektorjev je $d_1 = 1$.

$$B^2 = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \end{array}$$

$\operatorname{rang} B^2 = 1$
$\Rightarrow d_2 = 2$

Iščemo $v \in \ker B^2 \setminus \ker B$.

$-y + 2z = 0$
$y = 2, \ z = 1$

$v \in (0, 2, 1)$

$$Bv = (1,0,0) \notin \ker B \quad ☺$$

$\lambda = 2$:

$$B = A - 2I = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$-x + y - z = 0 \Rightarrow x = 0$$
$$-y + z = 0 \Rightarrow y = z$$

$$w = (0,1,1)$$

$$P = \begin{array}{c} \begin{matrix} Bv & v & w \end{matrix} \\ \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 2 & 1 \\ 0 & 1 & 1 \end{array}\right] \end{array}$$

$$P^{-1} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & -1 \\ 0 & -1 & 2 \end{array}\right]$$

$$f(J(A)) = \left[\begin{array}{cc|c} f(1) & f'(1) & \\ & f(1) & \\ \hline & & f(2) \end{array}\right] = \left[\begin{array}{cc|c} e^1 & e^1 & \\ & e^1 & \\ \hline & & e^2 \end{array}\right]$$

$$e^A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e & e & 0 \\ 0 & e & 0 \\ 0 & 0 & e^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} e & e & -e \\ 0 & 2e - e^2 & -2e + 2e^2 \\ 0 & e - e^2 & -e + 2e^2 \end{bmatrix}$$

---

3) $A \in \mathbb{C}^{n \times n}$

a) Dokaži: $\det e^{A} = e^{trA}$

$$A = P \cdot J(A) \cdot P^{-1}$$
$$e^{A} = P \cdot e^{J(A)} \cdot P^{-1}$$

$$e^{\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}} = \begin{bmatrix} e^{\lambda} & e^{\lambda} & \frac{e^{\lambda}}{2!} & \cdots \\ & \ddots & \ddots & \frac{e^{\lambda}}{2!} \\ & & \ddots & e^{\lambda} \\ 0 & & & e^{\lambda} \end{bmatrix}$$

Splošno:

$$f(J(A)) \overset{(1)}{=} \begin{bmatrix} f(\lambda_1) & & * \\ & \ddots & \\ 0 & & f(\lambda_n) \end{bmatrix}$$

$$\det e^{A} = \det(P \cdot e^{J(A)} \cdot P^{-1}) = \cancel{\det P} \cdot \det e^{J(A)} \cdot \cancel{\det P^{-1}}$$
$$\Rightarrow \det e^{A} = \det e^{J(A)}$$

$$\det e^{J(A)} \overset{(1)}{=} e^{\lambda_1} \cdot \ldots \cdot e^{\lambda_n} = e^{\lambda_1 + \ldots + \lambda_n}$$

$$A \sim J(A)$$
$$\Rightarrow trA = tr J(A)$$

$$e^{tr J(A)} = e^{\lambda_1 + \ldots + \lambda_n}$$

$$\Rightarrow \det e^{A} = e^{\lambda_1 + \ldots + \lambda_n} = e^{trA}$$

b) Dokaži: A se diagonalizira $\Leftrightarrow$ e^{A} se diagonalizira

A se diagonalizira $\Leftrightarrow$ celice v $J(A)$ so velikosti 1×1

$(\Rightarrow)$ $A \sim \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

$$f(A) \sim f(J(A)) = \begin{bmatrix} f(\lambda_1) & & & \\ & f(\lambda_2) & & \\ & & \ddots & \\ & & & f(\lambda_n) \end{bmatrix}$$

$f(A)$ se diagonalizira za vsako gladko funkcijo f

$(\Leftarrow)$ V splošnem ne velja:
$f(A)$ se diagonalizira $\not\Rightarrow$ A se diagonalizira

A se ne diagonalizira
$f(x) = 2$

$$\Rightarrow f(A) = \begin{bmatrix} 2 & & \\ & \ddots & \\ & & 2 \end{bmatrix} = 2I$$

$\Rightarrow$ $f(A)$ se diagonalizira, A pa se ne

Dokazujemo: Če se A ne diagonalizira, se tudi e^A ne diagonalizira.

Vsaj ena celica je večja od 1×1 $(k > 1)$.

$$J(A) = \begin{bmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_1 \end{matrix}} & \\ & \ddots \end{bmatrix}$$

$$e^{y(A)} = \begin{bmatrix} \boxed{\begin{matrix} e^{\lambda_1} & \lambda_1 & * \\ & \ddots & \lambda_1 \\ 0 & & e^{\lambda_1} \end{matrix}} & \\ & \ddots \end{bmatrix}$$

Spomnimo se:

$$y\begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_r \end{bmatrix} = \begin{bmatrix} y(B_1) & & & \\ & y(B_2) & & \\ & & \ddots & \\ & & & y(B_r) \end{bmatrix}$$

Pokažemo torej lahko, da se $y(B_1)$ ne diagonalizira.

$$B_1 = \begin{bmatrix} e^{\lambda_1} & \lambda_1 & & * \\ & e^{\lambda_1} & \ddots & \\ & & \ddots & \lambda_1 \\ 0 & & & e^{\lambda_1} \end{bmatrix}$$

$\lambda = e^{\lambda_1}$ je lastna vrednost matrike B_1

$$B_1 - \lambda I = \underbrace{\begin{bmatrix} 0 & e^{\lambda_1} & & * \\ & \ddots & \ddots & \\ & & \ddots & e^{\lambda_1} \\ 0 & & & 0 \end{bmatrix}}_{k} \; \begin{matrix} \left.\vphantom{\begin{matrix} 1\\1\\1 \end{matrix}}\right\} k-1 \\ \\ \end{matrix}$$

Prvih $k-1$ vrstic je linearno neodvisnih.
$\Rightarrow$ rang $(B_1 - \lambda I) = k-1$

$d_1 = \dim \ker (B_1 - \lambda I) = k - (k-1) = 1$
$\Rightarrow$ Število celic je 1.

$$J(B_1) = \begin{bmatrix} e^{\lambda_1} & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & e^{\lambda_1} \end{bmatrix}$$

$J(B_1)$ ni diagonalna

$\Rightarrow J(e^{J(A)})$ ni diagonalna

$e^A \sim e^{J(A)}$

$\Rightarrow e^A$ se ne diagonalizira :)

---

2) Naj bo $f(x) = \cos(\pi x + \frac{\pi}{4})$ in $A = \begin{bmatrix} -2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ -2 & 0 & 0 & -1 \end{bmatrix}$.

Izračunaj $f(A)$.

$$\Delta_A(\lambda) = \begin{vmatrix} -2-\lambda & 1 & -1 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 3 & 0 & 2-\lambda & 0 \\ -2 & 0 & 0 & -1-\lambda \end{vmatrix} = (-1-\lambda)\cdot \begin{vmatrix} -2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{vmatrix} =$$

$$= (\lambda+1)(\lambda-1)(\lambda^2-4+3) = (\lambda-1)^2(\lambda+1)^2$$

$\lambda = 1$: $B = A - I = \begin{bmatrix} -3 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & -2 \end{bmatrix}$

$\text{rang}\, B = 3$
$d_1 = 4 - 3 = 1$

$\Rightarrow$ ena kletka za $\lambda = 1$

$$B^2 = \begin{bmatrix} 6 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & 3 & -2 & 0 \\ 10 & -2 & 2 & 4 \end{bmatrix}$$

$\operatorname{rang} B^2 = 2$
$d_2 = 4 - 2 = 2$

...

$\Rightarrow$ velikost kletke je 2×2

Iščemo $v \in \ker B^2 \setminus \ker B$.

...

$v = (\frac{1}{2}, 1, 0, -\frac{3}{4})$
$Bv = (-\frac{1}{2}, 0, \frac{3}{2}, \frac{1}{2})$

$\lambda = -1$: $C = A + I = \begin{bmatrix} -1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$

$\operatorname{rang} C = 3$
$d_1 = 4 - 3 = 1$

$\Rightarrow$ ena kletka za $\lambda = 1$
$\Rightarrow$ velikost kletke je 2×2

$$C^2 = \begin{bmatrix} -2 & -1 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ 6 & 3 & 6 & 0 \\ 2 & -2 & 2 & 0 \end{bmatrix}$$

$$\operatorname{rang} C^2 = 2$$
$$d_2 = 4-2 = 2$$

Iščemo $w \in \ker C^2 \setminus \ker C$.

...

$$v = (-1, 0, 1, 0)$$
$$Cv = (0, 0, 0, 2)$$

$$J = \left[\begin{array}{cc|cc} 1 & 1 & & \\ & 1 & & \\ \hline & & -1 & 1 \\ & & & -1 \end{array}\right]$$

$$P = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & 1 & 0 & 0 \\ \frac{3}{2} & 0 & 0 & 1 \\ \frac{1}{2} & -\frac{3}{4} & 2 & 0 \end{bmatrix}$$

$$f(A) = P \cdot f(J) \cdot P^{-1}$$

$$f(J) = \left[\begin{array}{cc|cc} f(1) & f'(1) & & \\ & f(1) & & \\ \hline & & f(-1) & f'(-1) \\ & & & f(-1) \end{array}\right] = \left[\begin{array}{cc|cc} -\frac{\sqrt{2}}{2} & -\frac{\pi\sqrt{2}}{2} & & \\ & -\frac{\sqrt{2}}{2} & & \\ \hline & & \frac{-\sqrt{2}}{2} & \frac{-\pi\sqrt{2}}{2} \\ & & & \frac{-\sqrt{2}}{2} \end{array}\right]$$

$$f(A) = \dots$$