1) Preveri, da jo naslednje preslikave $\mathcal{A}: \mathbb{R}^2 \to \mathbb{R}^2$ linearne in opiši njihov geometrijski učinek.

U, V prostora nad $\mathbb{F}$

$\mathcal{A}: U \to V$ je linearna

$\Leftrightarrow \forall \alpha, \beta \in \mathbb{F}. \forall x, y \in U. \ \mathcal{A}(\alpha x + \beta y) = \alpha \mathcal{A}x + \beta \mathcal{A}y$

a) $\mathcal{A}(x,y) = (y,x)$

$\mathcal{A}(\alpha(x,y) + \beta(u,v)) = \mathcal{A}(\alpha x + \beta u, \alpha y + \beta v) = (\alpha y + \beta v, \alpha x + \beta u)$

$\alpha \mathcal{A}(x+y) + \beta \mathcal{A}(u,v) = \alpha(y,x) + \beta(v,u) = (\alpha y + \beta v, \alpha x + \beta u)$

geometrijsko: zrcaljenje čez $y = x$

b) $\mathcal{A}(x,y) = (x,0)$

$\mathcal{A}(\alpha x + \beta u, \alpha y + \beta v) = (\alpha x + \beta v, 0)$

$\alpha \mathcal{A}(x,y) + \beta \mathcal{A}(u,v) = \ldots = (\alpha x + \beta v, 0)$

geometrijsko: projekcija na $x = 0$

c) $\mathcal{A}(x,y) = (x, ay)$, $a \in \mathbb{R}$

$\mathcal{A}(\alpha x + \beta u, \alpha y + \beta v) = (\alpha x + \beta u, a(\alpha y + \beta v)) =$

$= \alpha(x, ay) + \beta(u, av) = \alpha \mathcal{A}(x,y) + \beta \mathcal{A}(u,v)$

geometrijsko: razteg za faktor a v smeri y-osi

d) $\mathcal{A}(x,y) = (x,x)$

$$\mathcal{A}(\alpha x+\beta u, \alpha y+\beta v) = (\alpha x+\beta u, \alpha x+\beta u) =$$
$$= \alpha(x,x)+\beta(u,u) = \alpha\mathcal{A}(x,y) + \beta\mathcal{A}(u,v)$$

geometrijsko: projekcija na $y=x$ v smeri vektorja $(0,1)$

Povzetek linearnih preslikav:
- zrcaljenje čez premico skozi izhodišče
- projekcija na premico skozi izhodišče v poljubni smeri
- razteg v poljubni smeri

---

3) Naj bosta X, Y vektorska prostora in $\mathcal{A}: X \to Y$ linearna preslikava. Naj bodo $x_1, \dots, x_n \in X$. Dokaži, da če so $\mathcal{A}x_1, \dots, \mathcal{A}x_n$ linearno neodvisni, so tudi $x_1, \dots, x_n$ linearno neodvisni.

$$\alpha_1 x_1 + \dots + \alpha_n x_n = 0 \Rightarrow \underline{\alpha_1 = \dots = \alpha_n = 0}$$

$$\alpha_1 x_1 + \dots + \alpha_n x_n = 0$$
$$\mathcal{A}(\alpha_1 x_1 + \dots + \alpha_n x_n) = \mathcal{A}(0)$$
$$\alpha_1 \mathcal{A} x_1 + \dots + \alpha_n \mathcal{A} x_n = 0$$

$\Rightarrow \alpha_1 = \dots = \alpha_n = 0$, ker so $\mathcal{A}x_1, \dots, \mathcal{A}x_n$ linearno neodvisni

$\Rightarrow x_1, \dots, x_n$ linearno neodvisni

---

4) U, W podprostora V

$\mathcal{A}: V \to V$ linearna preslikava

a) Dokaži: $\mathcal{A}(U+W) = \mathcal{A}(U) + \mathcal{A}(W)$

$U+W = \{u+w;\ u\in U,\ w\in W\}$

$\mathcal{A}(U+W) = \{\mathcal{A}(u+w);\ u\in U,\ w\in W\} = \{\mathcal{A}u + \mathcal{A}w;\ u\in U,\ w\in U\} =$
$= \{\mathcal{A}u;\ u\in U\} + \{\mathcal{A}w;\ w\in W\} = \mathcal{A}U + \mathcal{A}W$

b) Dokaži: $\mathcal{A}(U\cap W) \subseteq \mathcal{A}(U) \cap \mathcal{A}(W)$

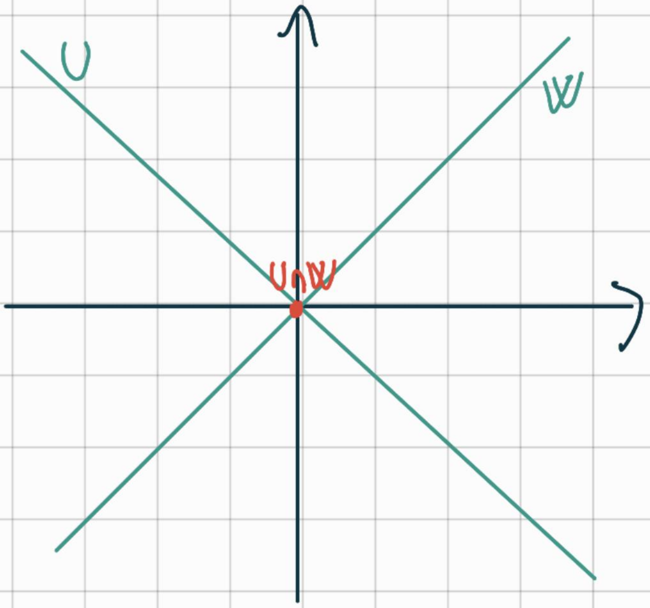
$\mathcal{A}(U\cap W) = \{\mathcal{A}x;\ x\in U\cap W\} = \{\mathcal{A}x;\ x\in U \wedge x\in W\}$

$y\in\mathcal{A}(U\cap W)$
$y = \mathcal{A}x,\ x\in U \wedge x\in V$

1) $x\in U \Rightarrow y\in\mathcal{A}(U)$
2) $x\in W \Rightarrow y\in\mathcal{A}(W)$

$\Rightarrow y\in\mathcal{A}(U)\cap\mathcal{A}(W)$

c) Najdi primer, ko v (b) enakost ne velja.



$U\cap W = \{0\} \Rightarrow \mathcal{A}(U\cap W) = \{0\}$
$\neq \mathcal{A}(U)\cap\mathcal{A}(W)$

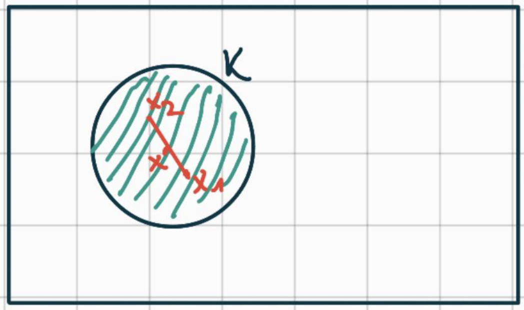
$\mathcal{A}$ poljubna projekcija na x-os
$\Rightarrow \mathcal{A}(U) = X,\ \mathcal{A}(W) = X$
$\Rightarrow \mathcal{A}(U)\cap\mathcal{A}(W) = X$

$\{0\} \neq X$

---

5) Naj bo $\mathcal{A}: X\to Y$ linearna preslikava. Dokaži, da je slika konveksne množice konveksna množica.

K je konveksna $\Leftrightarrow$ $(x_1, x_2 \in K \Rightarrow$ daljica $x_1 x_2 \subseteq K)$



$\mathcal{A}(K) \subseteq Y$ konveksna

Naj bo $y_1, y_2 \in \mathcal{A}(K)$

$y_1 = \mathcal{A}(x_1), \quad x_1 \in K$

$y_2 = \mathcal{A}(x_2), \quad x_2 \in K$

$x \in x_1 x_2$:

$x = x_1 + \alpha(x_2 - x_1), \quad \alpha \in [0, 1]$

$y \in y_1 y_2$:

$y = y_1 + \beta(y_2 - y_1), \quad \beta \in [0, 1]$

$y \in \mathcal{A}(K)$

$$\begin{aligned} y &= \mathcal{A}(x_1) + \beta(\mathcal{A}(x_2) - \mathcal{A}(x_1)) = \\ &= \mathcal{A}(x_1) + \beta(\mathcal{A}(x_2 - x_1)) = \\ &= \mathcal{A}(x_1) + \mathcal{A}(\beta(x_2 - x_1)) = \\ &= \mathcal{A}(x_1 + \beta(x_2 - x_1)) \end{aligned}$$

$x_1, x_2 \in K$, K je konveksna

$\Rightarrow x_1 + \beta(x_2 - x_1) \in K$, ker je K konveksna

$\Rightarrow y = \mathcal{A}(x) \in \mathcal{A}(K)$

7) Naj bosta $\vec{a}, \vec{b} \in \mathbb{R}^3$ linearno neodvisna.
Definiramo $\mathcal{A}: \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{x} \mapsto \vec{a} \times (\vec{b} \times \vec{x})$.

a) Pokaži, da je $\mathcal{A}$ linearna.
b) Določi jedro in rang preslikave $\mathcal{A}$.
c) Za kakšna $\vec{a}$ in $\vec{b}$ je $\mathcal{A}^m = \mathcal{A}$ za $m \in \mathbb{N}$, $m \geq 2$?

a) $$\begin{aligned}\mathcal{A}(\alpha\vec{x} + \beta\vec{y}) &= \vec{a} \times (\vec{b} \times (\alpha\vec{x} + \beta\vec{y})) = \\ &= \vec{a} \times (\vec{b} \times \alpha\vec{x} + \vec{b} \times \beta\vec{y}) = \\ &= \vec{a} \times (\vec{b} \times \alpha\vec{x}) + \vec{a} \times (\vec{b} \times \beta\vec{y}) = \\ &= \alpha\, \vec{a} \times (\vec{b} \times \vec{x}) + \beta\, \vec{a} \times (\vec{b} \times \vec{y}) = \alpha \mathcal{A}(x) + \beta \mathcal{A}(y)\end{aligned}$$

b) $\ker \mathcal{A} = \{x \in X;\ \mathcal{A}x = 0\}$

$$\vec{a} \times (\vec{b} \times \vec{x}) = 0$$

$$x = \alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a} \times \vec{b})$$

$$\vec{a} \times (\vec{b} \times (\alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a} \times \vec{b}))) = 0$$
$$\vec{a} \times (\vec{b} \times \alpha\vec{a} + b \times \gamma(\vec{a} \times \vec{b})) = 0$$
$$\alpha(\vec{a} \times (\vec{b} \times \vec{a})) + \gamma(\vec{a} \times (\vec{b} \cdot \vec{a}) \cdot \vec{b}) = 0$$
$$\alpha((\vec{a}\vec{a})\vec{b} - (\vec{a}\vec{b})\vec{a}) + \gamma(\vec{a}\vec{b})(\vec{a} \times \vec{b}) = 0$$

$\vec{a}$: $\quad \alpha(\vec{a} \cdot \vec{a}) = 0 \Rightarrow \alpha = 0$
$\vec{b}$: $\quad \alpha(\vec{a} \cdot \vec{b}) = 0$
$\vec{a} \times \vec{b}$: $\quad \gamma(\vec{a} \cdot \vec{b}) = 0 \Rightarrow \gamma = 0$ ali $\vec{a}\vec{b} = 0$

1) $\vec{a}\vec{b} \neq 0$:
$\alpha = 0$, $\gamma = 0$, $\beta \in \mathbb{R}$
$\vec{x} \in \ker \mathcal{A} \Leftrightarrow \vec{x} = \beta\vec{b}$, $\beta \in \mathbb{R}$
$\ker \mathcal{A} = \{\vec{b}\}$

2) $\vec{a}\vec{b}=0$:

$\alpha=0,\ \beta\in\mathbb{R},\ \gamma\in\mathbb{R}$

$\vec{x}\in\ker\mathcal{A} \iff \vec{x}=\beta\vec{b}+\gamma(\vec{a}\times\vec{b})$

$\ker\mathcal{A}=\{\vec{b},\vec{a}\times\vec{b}\}$

$\operatorname{rang}\mathcal{A}=\dim\operatorname{im}\mathcal{A}$

dimenzijska enačba:

$\mathcal{A}: X\to Y$

$\dim\operatorname{im}\mathcal{A}+\dim\ker\mathcal{A}=\dim X$

1) $\vec{a}\vec{b}\neq 0$:

$\operatorname{rang}\mathcal{A}=\dim X-\dim\ker\mathcal{A}=3-1=2$

2) $\vec{a}\vec{b}=0$:

$\operatorname{rang}\mathcal{A}=\dim X-\dim\ker\mathcal{A}=3-2=1$

c) $\mathcal{A}: X\to X$

$\mathcal{A}^m=\mathcal{A}\circ\mathcal{A}\circ\ldots\circ\mathcal{A}$

$$\mathcal{A}^2\vec{x}=(\mathcal{A}\circ\mathcal{A})(\vec{x})=\mathcal{A}(\mathcal{A}(\vec{x}))=\mathcal{A}(\vec{a}\times(\vec{b}\times\vec{x}))=$$
$$=\vec{a}\times(\vec{b}\times(\vec{a}\times(\vec{b}\times\vec{x})))=\vec{a}\times(\vec{b}\times((\vec{a}\vec{x})\vec{b}-(\vec{a}\vec{b})\vec{x}))=$$
$$=\vec{a}\times(-\vec{b}\times(\vec{a}\vec{b})\vec{x})=-(\vec{a}\vec{b})(\vec{a}\times(\vec{b}\times\vec{x}))=-(\vec{a}\vec{b})\mathcal{A}(x)$$

$\Rightarrow \mathcal{A}^2=-(\vec{a}\vec{b})\mathcal{A}$

1) $\vec{a}\vec{b}=-1$

$\Rightarrow \mathcal{A}^2=\mathcal{A}\quad(m=2)$

2) $\vec{a}\vec{b}\neq -1$

$\mathcal{A}^3=\mathcal{A}\circ\mathcal{A}^2=\mathcal{A}\circ(-(\vec{a}\vec{b})\mathcal{A})=-(\vec{a}\vec{b})(\mathcal{A}^2)=(-\vec{a}\vec{b})^2\mathcal{A}$

$\mathcal{A}^m=(-\vec{a}\vec{b})^{m-1}\mathcal{A}$

$\mathcal{A}^m=\mathcal{A}\iff(-\vec{a}\vec{b})^{m-1}=1\iff\vec{a}\vec{b}=\pm 1$

8) $\vec{a} \in \mathbb{R}^3,\ \vec{a} \neq 0$
$\vec{b} \in \mathbb{R}^3$

Dokaži, da ima enačba $\underbrace{(\vec{x}\cdot\vec{a})\cdot\vec{a} + \vec{x}\times\vec{a}}_{\mathcal{A}x} = \vec{b}$ natanko eno rešitev.

A je linearna preslikava

$$\mathcal{A}(\alpha\vec{x}+\beta\vec{y}) = ((\alpha\vec{x}+\beta\vec{y})\cdot\vec{a})\cdot\vec{a} + (\alpha\vec{x}+\beta\vec{y})\times\vec{a} =$$
$$= (\alpha\vec{x}\vec{a} + \beta\vec{y}\vec{a})\vec{a} + \alpha\vec{x}\times\vec{a} + \beta\vec{y}\times\vec{a} =$$
$$= \alpha(\vec{x}\vec{a})\vec{a} + \beta(y\vec{a})\vec{a} + \alpha\vec{x}\times\vec{a} + \beta\vec{y}\times\vec{a} =$$
$$= \alpha(\vec{x}\vec{a})\vec{a} + \alpha\vec{x}\times\vec{a} + \beta(y\vec{a})\vec{a} + \beta y\times\vec{a} =$$
$$= \alpha\mathcal{A}\vec{x} + \beta\mathcal{A}\vec{y}$$

$\mathcal{A}\vec{x} = \vec{b}$ ima natanko eno rešitev

$\mathcal{A}$ je injektivna $\Leftrightarrow$ $\ker\mathcal{A} = \{0\}$

$$\mathcal{A}\vec{x} = 0$$
$$(\vec{x}\vec{a})\vec{a} + \vec{x}\times\vec{a} = 0$$

$$(\vec{x}\vec{a})\vec{a} \perp \vec{x}\times\vec{a}$$
$$\Rightarrow (\vec{x}\vec{a})\vec{a} = \vec{x}\times\vec{a} = 0$$
$$\Rightarrow \vec{x}\cdot\vec{a} = 0 \quad \wedge \quad \vec{x} = \alpha\vec{a}$$
$$\Rightarrow \vec{x} = 0$$

$$\Rightarrow \ker\mathcal{A} = \{0\}$$

$\mathcal{A}$ je surjektivna

$\mathcal{A}: X \to X$, $\dim X < \infty$, $\ker \mathcal{A} = \{0\}$
$\Rightarrow \mathcal{A}$ je bijektivna

Naša preslikava je bijektivna, zato ima enačba natanko eno rešitev.

---

9) $\mathcal{A}: \mathbb{R}_n[x] \to \mathbb{R}_n[x]$
$p(x) \longmapsto p(x+1) - p(x)$

Dokaži, da je $\mathcal{A}$ linearna in določi $\ker \mathcal{A}$ in $\operatorname{im} \mathcal{A}$.

Linearnost:

$$\begin{aligned}
\mathcal{A}(\alpha p + \beta q)(x) &= (\alpha p + \beta q)(x+1) - (\alpha p + \beta q)(x) = \\
&= (\alpha p)(x+1) - (\alpha p)(x) + (\beta q)(x+1) - (\beta q)(x) = \\
&= \alpha(p(x+1) - p(x)) + \beta(q(x+1) - q(x)) = \\
&= \alpha \mathcal{A}(p)(x) + \beta \mathcal{A}(q)(x) = \\
&= (\alpha \mathcal{A} p + \beta \mathcal{A} q)(x)
\end{aligned}$$

$\ker \mathcal{A}$:

$$p \in \ker \mathcal{A} \Leftrightarrow \mathcal{A}p = 0 \Leftrightarrow (\forall x. (\mathcal{A}p)(x) = 0) \Leftrightarrow (\forall x.\ p(x+1) = p(x))$$

Nekonstanten polinom ni periodična funkcija:
$p(x+a) = p(x) \quad \forall x,\ a > 0$
p je določen z vrednostmi na $[0,a]$
$p: [0,a] \xrightarrow{\text{zveznost}} \mathbb{R}$ je omejena
$\cancel{\quad} \quad \lim_{x \to \infty} p(x) = \pm \infty$

$p \in \ker \mathcal{A} \Leftrightarrow p$ je konstanten

$B_{\ker A} = \{1\}$
$\dim \ker A = 1$

$\operatorname{im} A$:

$\dim \operatorname{im} A = \dim \mathbb{R}_n[x] - \dim \ker A = n+1 - 1 = n$

ogrodje slike:
$\{A1, Ax, Ax^2, \dots, Ax^n\}$

$(A1)(x) = 0$
$(Ax)(x) = x+1-x = 1$
$(Ax^2)(x) = (x+1)^2 - x^2 = 2x+1$
$\vdots$
$(Ax^k)(x) = (x+1)^k - x^k = k \cdot x^{k-1} + \dots$
$\vdots$
$(Ax^n)(x) = (x+1)^n - x^n = n \cdot x^{n-1} + \dots$

Ti polinomi so neodvisni, ker so različnih stopenj.
Ker jih je n in so tudi ogrodje, so baza slike.

$\operatorname{im} A = \mathbb{R}_{n-1}[x]$

---

12) Naj bo $A: V \to V$ linearna preslikava.

a) Dokaži: $\ker A^n \subseteq \ker A^{n+1}$ $\quad \forall n \in \mathbb{N}$
b) Dokaži: $\operatorname{im} A^n \supseteq \operatorname{im} A^{n+1}$ $\quad \forall n \in \mathbb{N}$
c) Dokaži: $A^2 = 0 \iff \operatorname{im} A \subseteq \ker A$
d) Dokaži: $\operatorname{im} A \cap \ker A = \{0\} \Rightarrow \ker A^n = \ker A$ $\quad \forall n \in \mathbb{N}$

a) Naj bo $x \in \ker \mathcal{A}^n \Leftrightarrow \mathcal{A}^n x = 0$

Dokazujemo $x \in \ker \mathcal{A}^{n+1} \Leftrightarrow \mathcal{A}^{n+1} x = 0$

ker je $\mathcal{A}$ linearna

$$\mathcal{A}^{n+1} x = (\mathcal{A} \circ \mathcal{A}^n) x = \mathcal{A}(\mathcal{A}^n x) = \mathcal{A}(0) = 0$$

b) Naj bo $x \in \operatorname{im} \mathcal{A}^{n+1} \Leftrightarrow x = \mathcal{A}^{n+1}(y)$

Dokazujemo $x \in \mathcal{A}^n \Leftrightarrow x = \mathcal{A}^n(z)$

$$x = \mathcal{A}^{n+1}(y) = \mathcal{A}^n(\mathcal{A}(y)) = \mathcal{A}^n(z)$$

c) ($\Rightarrow$) $\mathcal{A}^2 = 0 \Rightarrow \operatorname{im} \mathcal{A} \subseteq \ker \mathcal{A}$

Naj bo $x \in \operatorname{im} \mathcal{A} \Leftrightarrow x = \mathcal{A} y$

Dokazujemo $x \in \ker \mathcal{A} \Leftrightarrow \mathcal{A} x = 0$

$$\mathcal{A}(x) = \mathcal{A}(\mathcal{A}(y)) = \mathcal{A}^2 y = 0$$

($\Leftarrow$) $\operatorname{im} \mathcal{A} \subseteq \ker \mathcal{A} \Rightarrow \mathcal{A}^2 = 0$

$$\mathcal{A}^2 y = \mathcal{A}(\overbrace{\mathcal{A} y}^{\in \operatorname{im} \mathcal{A} \subseteq \ker \mathcal{A}}) = 0$$

$\Rightarrow \mathcal{A}^2 = 0$

d) $\ker \mathcal{A} \subseteq \ker \mathcal{A}^n$:

Po točki (a) je $\ker \mathcal{A} \subseteq \ker \mathcal{A}^2 \subseteq \ldots \subseteq \ker \mathcal{A}^n$

$\Rightarrow \ker \mathcal{A} \subseteq \ker \mathcal{A}^n$

$\ker \mathcal{A}^n \subseteq \ker \mathcal{A}$:

$n=2$: $\mathcal{A}(\mathcal{A} x) = 0 \Rightarrow \mathcal{A} x \in \ker \mathcal{A}$ in $\mathcal{A} x \in \operatorname{im} \mathcal{A}$

$\Rightarrow \mathcal{A} x \in \ker \mathcal{A} \cap \operatorname{im} \mathcal{A} = \{0\}$

$\Rightarrow x \in \ker \mathcal{A}$ ✓

$n \to n+1$: I.P.: $\ker \mathcal{A}^n \subseteq \ker \mathcal{A}$

CIW: $\ker \mathcal{A}^{n+1} \subseteq \ker \mathcal{A}$

$\mathcal{A}^{n+1}(x) = \mathcal{A}^n(\mathcal{A}x) = 0$
$\Rightarrow \mathcal{A}x \in \ker \mathcal{A}^n$
$\Rightarrow$ po I.P. $\ker \mathcal{A}^n = \ker \mathcal{A}$
$\Rightarrow \mathcal{A}x \in \ker \mathcal{A}$ in $\mathcal{A}x \in \operatorname{im} \mathcal{A}$
$\Rightarrow \mathcal{A}x \in \ker \mathcal{A} \cap \operatorname{im} \mathcal{A} = \{0\}$
$\Rightarrow x \in \ker \mathcal{A}$ ✓

---

13) $V = U \oplus W$

a) Dokaži, da obstaja natanko ena linearna preslikava $\mathcal{P}: V \to V$ (projektor na U vzdolž W), da je $\mathcal{P}u = u$ za vsak $u \in U$ in $\mathcal{P}w = 0$ za vsak $w \in W$.

b) Dokaži: $\ker \mathcal{P} = W$, $\operatorname{im} \mathcal{P} = U$

c) Dokaži: $\mathcal{P}^2 = \mathcal{P}$

d) Dokaži: Q je projektor na W vzdolž U
$\Rightarrow \mathcal{P} \circ Q = Q \circ \mathcal{P} = 0$ in $\mathcal{P} + Q = id$

a) Naj bo $v \in V$: $\mathcal{P}v = \mathcal{P}(\underbrace{u + w}_{\text{enolično}}) = \mathcal{P}(u) + \mathcal{P}(w) = u + 0 = u$

Preverimo linearnost:
$$\mathcal{P}(\alpha v_1 + \beta v_2) = \mathcal{P}(\alpha(u_1 + w_1) + \beta(u_2 + w_2)) =$$
$$= \mathcal{P}(\alpha u_1 + \beta u_2 + \alpha w_1 + \beta w_2) =$$
$$= \alpha u_1 + \beta u_2 \overset{?}{=}$$
$$\overset{?}{=} \alpha \mathcal{P} v_1 + \beta \mathcal{P} v_2 =$$
$$= \alpha \mathcal{P}(u_1 + w_1) + \beta \mathcal{P}(u_2 + w_2) =$$
$$= \alpha u_1 + \beta u_2$$

b) $v \in \ker \mathcal{P} \Leftrightarrow \mathcal{P}v = 0 \Leftrightarrow \mathcal{P}(u + w) = 0 \Leftrightarrow u = 0 \Leftrightarrow v \in W$
$y \in \operatorname{im} \mathcal{P} \Leftrightarrow y = \mathcal{P}v \Leftrightarrow y = \mathcal{P}(u + w) = u \Leftrightarrow y \in U$

c) $P^2 = P$

$P(Pv) = Pv$

$Pu \overset{?}{=} u$

$u = u$ ✓

d) $Qv = Q(u+w) = w$

$(P \circ Q)(v) = P(Q(v)) = P(Q(u+w)) = P(w) = 0$

$(Q \circ P)(v) = Q(P(v)) = Q(P(u+w)) = Q(u) = 0$

$(P+Q)(v) = P(u+w) + Q(u+w) = u + w = v$

---

2) Za $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ so dane linearne preslikave $A, B, C : \mathbb{R}^3 \to \mathbb{R}^3$ s predpisi $A\vec{x} = \vec{a} \times \vec{x}$, $B\vec{x} = \vec{b} \times \vec{x}$ in $C\vec{x} = \vec{c} \times \vec{x}$. Dokaži, da so preslikave A, B, C linearno neodvisne natanko tedaj, ko so vektorji $\vec{a}, \vec{b}, \vec{c}$ linearno neodvisni.

$(\Leftarrow)$ $\alpha A\vec{x} + \beta B\vec{x} + \gamma C\vec{x} = 0$

$\alpha(\vec{a} \times \vec{x}) + \beta(\vec{b} \times \vec{x}) + \gamma(\vec{c} \times \vec{x}) = 0$

$(\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}) \times \vec{x} = 0$

$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$

$\vec{a}, \vec{b}, \vec{c}$ lin. neodvisni

$\alpha = \beta = \gamma = 0$

$(\Rightarrow)$ $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$

$(\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}) \times \vec{x} = 0$

$\alpha A + \beta B + \gamma C = 0$

A, B, C lin. neodvisne

$\alpha = \beta = \gamma = 0$

---

6) Naj bo $\vec{a} \in \mathbb{R}^3$ in $A: \mathbb{R}^3 \to \mathbb{R}^3$ preslikava s predpisom $A\vec{x} = (\vec{x}\cdot\vec{a})\vec{a} + \vec{x}\times\vec{a} - 2\vec{x}$. Pokaži, da je A linearna in poišči njeno jedro v primeru, ko je $\vec{a} = (1,0,1)$.

$$A(\alpha\vec{x}+\beta\vec{y}) = ((\alpha\vec{x}+\beta\vec{y})\cdot\vec{a})\cdot\vec{a} + (\alpha\vec{x}+\beta\vec{y})\times\vec{a} - 2(\alpha\vec{x}+\beta\vec{y}) =$$
$$= (\alpha\vec{x}\vec{a} + \beta\vec{y}\vec{a})\vec{a} + \alpha\vec{x}\times\vec{a} + \beta\vec{y}\times\vec{a} - 2\alpha\vec{x} - 2\beta\vec{y}$$
$$= \underline{\alpha(\vec{x}\vec{a})\vec{a}} + \underline{\beta(\vec{y}\vec{a})\vec{a}} + \underline{\alpha\vec{x}\times\vec{a}} + \underline{\beta\vec{y}\times\vec{a}} \underline{- 2\alpha\vec{x}} \underline{- 2\beta\vec{y}}$$

$$\alpha A\vec{x} + \beta A\vec{y} = \alpha((x\vec{a})\vec{a} + \vec{x}\times\vec{a} - 2\vec{x}) + \beta((y\vec{a})\vec{a} + \vec{y}\times\vec{a} - 2\vec{y}) =$$
$$= \underline{\alpha(x\vec{a})\vec{a}} + \underline{\alpha\vec{x}\times\vec{a}} \underline{- 2\alpha\vec{x}} + \underline{\beta(\vec{y}\vec{a})\vec{a}} + \underline{\beta\vec{y}\times\vec{a}} \underline{- 2\beta\vec{y}}$$

$$\Rightarrow A(\alpha\vec{x}+\beta\vec{y}) = \alpha A\vec{x} + \beta A\vec{y}$$

$$\ker A = \{\vec{x} \in \mathbb{R}^3 ; A\vec{x} = 0\}$$

$$((x_1,x_2,x_3)(1,0,1))(1,0,1) + (x_1,x_2,x_3)\times(1,0,1) - 2(x_1,x_2,x_3) = 0$$
$$(x_1+x_3)(1,0,1) + (x_2, x_3-x_1, -x_2) - 2(x_1,x_2,x_3) = 0$$
$$(x_1+x_3+x_2-2x_1,\ x_3-x_1-2x_2,\ x_1+x_3-x_2-2x_3) = 0$$
$$(-x_1+x_2+x_3,\ -x_1-2x_2+x_3,\ x_1-x_2-x_3) = 0$$

$$-x_1+x_2+x_3 = 0 \Rightarrow x_1 = x_2+x_3$$
$$-x_1-2x_2+x_3 = 0 \Rightarrow x_1 = -2x_2+x_3$$

$$x_2+x_3 = -2x_2+x_3$$
$$x_2 = -2x_2$$
$$x_2 = 0$$

$$x_1 = x_3$$

$$\Rightarrow x = (x_1, 0, x_1) \in \ker A$$

---

10) Dana je preslikava $A : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ s predpisom $(Ap)(x) = p(1)(x^2+x+1) + \lambda p(x)$ za $p \in \mathbb{R}_2[x]$. Pokaži, da je A linearna in določi jedro v odvisnosti od $\lambda \in \mathbb{R}$.

$$A(\alpha p+\beta q)(x) = (\alpha p+\beta q)(x)\cdot(x^2+x+1)+\lambda(\alpha p+\beta q)(x) =$$
$$= \alpha\cdot p(x)\cdot(x^2+x+1) + \alpha\cdot\lambda\cdot p(x) + \beta\cdot q(x)\cdot(x^2+x+1) + \beta\cdot\lambda\cdot q(x) =$$
$$= \alpha A p(x) + \beta A q(x)$$

$$p \in A \iff p(1)(x^2+x+1) + \lambda p(x) = 0$$

$$(ax^2+bx+c)(1)\cdot(x^2+x+1) + \lambda(ax^2+bx+c) = 0$$
$$(a+b+c)(x^2+x+1) + \lambda a x^2 + \lambda b x + \lambda c = 0$$
$$ax^2+ax+a+bx^2+bx+b+cx^2+cx+c+\lambda a x^2+\lambda b x+\lambda c = 0$$
$$(a+b+c+\lambda a)x^2 + (a+b+c+\lambda b)x + (a+b+c+\lambda c) = 0$$

$$\begin{array}{lll} x^2: & a+b+c+\lambda a = 0 & \lambda(a-b)=0 \\ x: & a+b+c+\lambda b = 0 & \lambda(b-c)=0 \\ 1: & a+b+c+\lambda c = 0 & \end{array}$$

1.) $\lambda = 0$:

$a+b+c=0$

$c=-a-b$

$p(x) = ax^2+bx-a-c$

$p(x) = a(x^2-1)+b(x-1)$

2.) $\lambda \neq 0$:

$a=b=c$

$3a+\lambda a = 0$

$(3+\lambda)a = 0$

2.1) $\lambda = -3$:

$a \in \mathbb{R}$

$p(x) = ax^2+ax+a$

2.2) $\lambda \neq 3$:

$a = 0$

$p(x) = 0$

---

11) Ugotovi, katere preslikave so linearne. Za tiste, ki so, ugotovi, če so injektivne ali surjektivne.

$\mathcal{A}, \mathcal{B}, \mathcal{C}: C[0,1] \to C[0,1]$

a) $(\mathcal{A}f)(x) = f(x)^2$

$$\mathcal{A}(\alpha f + \beta g)(x) = ((\alpha f + \beta g)(x))^2$$
$$\alpha \mathcal{A}f(x) + \beta \mathcal{A}g(x) = \alpha f(x)^2 + \beta g(x)^2$$

ni linearna

b) $(\mathcal{B}f)(x) = e^{-x} f(x)$

$$\mathcal{B}(\alpha f + \beta g)(x) = e^{-x} \cdot (\alpha f + \beta g)(x) = e^{-x} \cdot (\alpha f(x) + \beta g(x)) =$$
$$= e^{-x} \alpha \cdot f(x) + e^{-x} \cdot \beta \cdot g(x) = \alpha \cdot \mathcal{B}f(x) + \beta \cdot \mathcal{B}f(x)$$

je linearna

injektivnost:

$$(\mathcal{B}f)(x) = (\mathcal{B}g)(x)$$
$$e^{-x} f(x) = e^{-x} g(x)$$
$$f(x) = g(x)$$

$$f = g$$

je injektivna

surjektivnost:

Ker je endomorfizem $C[0,1] \to C[0,1]$ in je injektivna, je po izreku tudi surjektivna.

c) $(Cf)(x) = f(x) + x$

$$C(\alpha f + \beta g)(x) = (\alpha f + \beta g)(x) + x = \alpha f(x) + \beta g(x) + x$$
$$\alpha C f(x) + \beta C g(x) = \alpha f(x) + \alpha x + \beta g(x) + \beta x$$

ni linearna

---

14) Naj bosta f in g neničelna linearna funkcionala na $\mathbb{R}^n$. Pokaži, da sta lin. odvisna natanko tedaj, ko imata isto jedro.

($\Rightarrow$) Predpostavimo: f, g lin. odvisna: $f = \lambda g$
Dokazujemo: $\ker f = \ker g$

$$\ker f = \{x \in \mathbb{R}^n ; f(x) = 0\}$$
$$\ker g = \{x \in \mathbb{R}^n ; g(x) = 0\} = \{x \in \mathbb{R}^n ; \lambda g(x) = 0\}$$

$$f(x) = 0 \Leftrightarrow \lambda f(x) = 0$$
$$\Rightarrow \ker f = \ker g$$

($\Leftarrow$) Predpostavimo: $\ker f = \ker g$
Dokazujemo: f, g lin. odvisna

Če imata f in g isto jedro, imata vse ničle enake
Torej sta linearno odvisna